Math 260 Section 2.4 and 2.5

Most important ideas:

- **Block matrices (also called partitioned matrices)**
- **LU decomposition of a matrix**
- **Divide and conquer is often a good idea—see Numerical Note on page 120. Also see Numerical Note on page 127.**

Example 1: A diagonal matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$ has inverse $\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

In general, the inverse of a block diagonal matrix $\begin{bmatrix} \boldsymbol{A} & 0 & 0 \\ 0 & \boldsymbol{B} & 0 \\ 0 & 0 & \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A^{-1}} & 0 & 0 \\ 0 & \boldsymbol{B^{-1}} & 0 \\ 0 & 0 & \boldsymbol{C^{-1}} \end{bmatrix}$.

Example 2: The inverse of $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 3/2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Many real-life matrices are block diagonal, with each block being the same size. (And we can partition the matrix into its blocks to make it easier to work with.)

Problem 1: Given the blocks of a matrix A, B, C, find X, Y, Z so that $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ X & Y \end{bmatrix} = \begin{bmatrix} 0 & I \\ Z & 0 \end{bmatrix}$.
What we need to have happen and the result of each (A, B, C, I, X, Y, Z and 0 are all matrices):

$\boldsymbol{AI + BX = 0 \Rightarrow BX = -A \Rightarrow X = -B^{-1}A}$
$\boldsymbol{A0 + BY = I \Rightarrow Y = B^{-1}}$
$\boldsymbol{CI + 0X = Z \Rightarrow Z = C}$
$\boldsymbol{C0 + 0Y = 0}$ **(which tells us nothing)**

Example of Problem 1: $\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 7 & 2 \\ 4 & 5 & 6 & 3 & 1 \\ \hline 2 & 0 & 3 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline ? & ? & ? & 0 & 0 \end{array}\right]$

So for this problem $\boldsymbol{A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}}$.

Then we find that the bottom left part of the second matrix is

$$\boldsymbol{X = -\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ -25 & -29 & -33 \end{bmatrix}},$$

the bottom right part of the second matrix is $\boldsymbol{Y = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}}$,

the bottom left part of the third matrix is $\boldsymbol{Z = C = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}}$.

You can check that $\boldsymbol{\begin{bmatrix} 1 & 2 & 3 & 7 & 2 \\ 4 & 5 & 6 & 3 & 1 \\ 2 & 0 & 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 7 & 8 & 9 & 1 & -2 \\ -25 & -29 & -33 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 3 & 0 & 0 \end{bmatrix}}$

Unlike this example, the blocks in many (most?) real-life matrices are square and of the same size.
Question: For which of the following systems of equations is it easier to find a solution? Why?

$$\begin{array}{rcl} 2x_1+2x_2+3x_3 &=& 7 \\ 5x_1-2x_2+4x_3 &=& 0 \\ 3x_1+\ \ x_2+5x_3 &=& 10 \end{array} \quad \text{or} \quad \begin{array}{rcl} 2x_1+2x_2+3x_3 &=& 7 \\ -2x_2+4x_3 &=& 10 \\ 5x_3 &=& 15 \end{array} \quad \text{or} \quad \begin{array}{lcr} 2x_1 &=& -4 \\ 5x_1-2x_2 &=& -12 \\ 3x_1+\ \ x_2+5x_3 &=& 10 \end{array}$$

Observation: upper triangular and lower triangular systems are easier to solve.

What if we could transform $A\vec{x}=\vec{b}$ into an upper or lower triangular system (or, as it turns out, a combination of both)? Suppose we could somehow factor A into a lower triangular (with zeros above the diagonal) matrix L and an upper triangular (with zeros below the diagonal) matrix U: $A=LU$. If we had $LU\vec{x}=\vec{b}$, where L is lower triangular and U is upper triangular, then we could do the following:

Let $\vec{y}=U\vec{x}$. Then $LU\vec{x}=\vec{b}$ is the same as $L\vec{y}=\vec{b}$. So solve for $\vec{y}$ in $L\vec{y}=\vec{b}$, then solve for $\vec{x}$ in $U\vec{x}=\vec{y}$.

Problem 2: Suppose $A=\begin{bmatrix}2&-1&1\\4&1&2\\-2&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}2&-1&1\\0&3&0\\0&0&1\end{bmatrix}=LU$ and where $\vec{b}=\begin{bmatrix}6\\15\\-3\end{bmatrix}$,

we want to solve for $\vec{x}$ in $A\vec{x}=\vec{b}$. We first solve for $\vec{y}$ in $L\vec{y}=\vec{b}$ using forward substitution, working from the top equation $y_1=6$ down to the bottom equation:

$$\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}6\\15\\-3\end{bmatrix}, \text{ that is, } \begin{array}{rcl} y_1 &=& 6 \\ 2y_1+y_2 &=& 15 \Rightarrow y_2=15-2(6)=3 \\ -y_1+y_3 &=& -3 \Rightarrow y_3=-3+6=3 \end{array}$$

We then solve for $\vec{x}$ in $U\vec{x}=\vec{y}$ using backward substitution, working from the bottom equation $x_3=3$ up to the top equation:

$$\begin{bmatrix}2&-1&1\\0&3&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}6\\3\\3\end{bmatrix}, \text{ that is, } \begin{array}{rcl} 2x_1-x_2+x_3 &=& 6 \Rightarrow x_1=\frac{1}{2}(6+1-3)=2 \\ 3x_2 &=& 3 \Rightarrow x_2=1 \\ x_3 &=& 3 \end{array}$$

So how do we find L and U for a given matrix A? We can transform via elementary row operations (which will do using elementary matrices—woo hoo!) A into an upper triangular matrix $E_n\cdots E_3E_2E_1A=U$, where U is simply the echelon form of A, and where

$$L=(E_n\cdots E_3E_2E_1)^{-1}=E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_n^{-1}.$$

Then we have $A=LU$, where L is lower triangular.

Problem 3: Transform A from Problem 2 into an upper triangular matrix (its echelon form), and in the process find the LU factorization of A.

First we add -2 times row 1 to row 2: $\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&-1&1\\4&1&2\\-2&1&0\end{bmatrix}=\begin{bmatrix}2&-1&1\\0&3&0\\-2&1&0\end{bmatrix}$

Next we add 1 times row 1 to row 3: $\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}2&-1&1\\0&3&0\\-2&1&0\end{bmatrix}=\begin{bmatrix}2&-1&1\\0&3&0\\0&0&1\end{bmatrix}$

So in all we have $\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&-1&1\\4&1&2\\-2&1&0\end{bmatrix}=\begin{bmatrix}2&-1&1\\0&3&0\\0&0&1\end{bmatrix}$**, that is, $E_2E_1A=U$.**

So $A=(E_2E_1)^{-1}U=LU$ where $L=(E_2E_1)^{-1}=E_1^{-1}E_2^{-1}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}$.

Note: the product of lower triangular matrices is lower triangular (and similarly for upper triangular). It turns out mathematicians use the LU factorization of A to do/find many of the things to/about A that we might want to do/find.