Most important ideas:

- **Recall why we like the identity matrix.**
- **Matrix inverse: what are they, how are they useful (Theorem 5, page 104), how to find them (Theorem 7 on page 107 and Algorithm on page 108), and do all matrices have inverses, and what does it mean if a matrix doesn't have an inverse?**
- **Formula for a 2×2 matrix on page 103 in Theorem 4.**
- **Rules/properties of inverses in Theorem 6 on page 105.**
- **Elementary matrices.**
- **Real life: see Numerical Note on page 109.**

First, how is a matrix inverse is useful, and why do we like the identity matrix so gosh darn much?

Compare solving for number x in $ax = b$ to solving for vector $\vec{x}$ in $A\vec{x} = \vec{b}$:

$$\begin{aligned} ax &= b \\ a^{-1}ax &= a^{-1}b \\ 1x &= a^{-1}b \\ x &= a^{-1}b \end{aligned} \qquad \begin{aligned} A\vec{x} &= \vec{b} \\ A^{-1}A\vec{x} &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

Problem 1: Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and use it to find the solution to $\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 6 \end{aligned}$.

$A^{-1} = \frac{1}{1\cdot 4 - 2\cdot 3}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$. **(Notice that $A^{-1}A = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.)**

Then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\vec{b} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$. Check that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -4 \\ 9/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Yup!

For larger than 2×2 matrices, how do you find the inverse? We'll answer this for now by still considering a 2×2 case. Recall that we can solve $A\vec{x} = \vec{b}$ for more than one right hand side $\vec{b}$.

Let's solve for $\vec{x}$ in $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with two right hand sides $\vec{b}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 3 & 4 & 5 & -2 \end{array}\right] \underset{\sim}{R2 + (-3)R1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 0 & -2 & 2 & 4 \end{array}\right] \underset{\sim}{-\tfrac{1}{2}R2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 0 & 1 & -1 & -2 \end{array}\right] \underset{\sim}{R1 + (-2)R2} \left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

So : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ **and** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$**, that is** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 5 & -2 \end{bmatrix}$.

We could repeat this, but now with the right hand sides of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

So : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **and** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$**, that is** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

That is, we've just discovered that the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$.

In general we find A^{-1} by doing the row reduction $[\,A \mid I\,] \sim [\,I \mid A^{-1}]$.

Problem 2: Find the inverse of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. **We will do $[\,A \mid I\,] \sim [\,I \mid A^{-1}]$:**

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

Check that $A^{-1}A = I$ (or $AA^{-1} = I$): $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **Yup!**

Problem 3: Find a 2×2 matrix that has no inverse, and write the system of equations (2 equations, 2 unknowns) corresponding to it. **Here is one. Try to come up with your own.**

$A = \begin{bmatrix} 2 & 5 \\ -6 & -15 \end{bmatrix}$ which has corresponding equations $\begin{array}{r} 2x_1 + 5x_2 = \textit{some number} \\ -6x_1 - 15x_2 = \textit{some number} \end{array}$

These equations with two (of many possible) right hand sides:

$\begin{array}{r} 2x_1 + 5x_2 = 1 \\ -6x_1 - 15x_2 = 2 \end{array}$ **which has no solution, or** $\begin{array}{r} 2x_1 + 5x_2 = 3 \\ -6x_1 - 15x_2 = -9 \end{array}$ **which has infinite solutions.**

In general, if A has an inverse, then $A\vec{x} = \vec{b}$ has **1** solution: $\vec{x} = A^{-1}\vec{b}$

If A does not have an inverse, then $A\vec{x} = \vec{b}$ has either **0** or ∞ solutions, depending on what $\vec{b}$ is.

Recall elementary matrices—a few examples:

Row operation	Elementary matrix
Add 4 · Row 1 to Row 2	$E_1 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$
Swap Row 1 and Row 2	$E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Multiply Row 1 by 5	$E_3 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

The elementary matrices that change A to I give us the inverse of A.

If $E_n \cdots E_2E_1A = I$, then A^{-1} must simply be $E_n \cdots E_2E_1$ since $E_n \cdots E_2E_1$ times A is I.
(This is what it means to be the inverse of A!)

Problem 4: Find inverse of $A = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$. **So what do we need to do to transform A to I?**

Next action	Elementary Matrix	Product and result
$\frac{1}{2}R1$	$E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$	$E_1A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$
$R2 - 3R1$	$E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$	$E_2(E_1A) = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$
$-R2$	$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
$R1 - R2$	$E_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$	$E_4(E_3E_2E_1A) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3/2 & -1 \end{bmatrix}$. **(Check!)**

The fact that A^{-1} is the product of elementary matrices for transforming A into I is very useful *theoretically*, but not for actually computing the inverse.