Most important ideas:

- **Rules and properties of matrix multiplication in Theorem 1, Theorem 2 and Theorem 3. Also see warnings on page 98.**
- **Very important: what multiplying one matrix by another really means: page 95.**
- **A handy way to compute matrix product on page 96.**

Problem 1: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$. Find $A\left[\,\vec{b}_1\,\vec{b}_2\,\right]$.

Use definition of matrix multiplication on page 110 and definition of matrix-vector multiplication on page 41 to find AB.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 8 \\ 11 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 14 \\ 20 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 5 \\ 8 \\ 11 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ 30 \\ 42 \end{bmatrix}.$$

So $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 8 & 18 \\ 14 & 30 \\ 20 & 42 \end{bmatrix}.$$

Notice dimensions: $4 \times 3 \;\; * \;\; 3 \times 2 \; = \; 4 \times 2$. In general: $a \times b \;\; * \;\; b \times c \; = \; a \times c$.

We could also use alternate rule for matrix multiplication on page 96 to find AB.

So, for example, to get the value 30 in row 3, column 2 we multiple row 3 of the first matrix with column 2 of the second matrix: $7 \cdot 3 + 8 \cdot 0 + 9 \cdot 1 = 30$.

Examples of some matrices we'll see in the next section:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}.$$

$$E_2A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}.$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 12 & 15 & 18 \\ 7 & 8 & 9 \end{bmatrix}.$$

These sorts of matrices accomplish the three elementary row operations: (1) add/subtract a multiple of one row to/from another; (2) swap two rows; (3) multiply a row by a number/scalar.

Problem 2: Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = [\,\vec{b}_1\ \vec{b}_2\,] = \begin{bmatrix} 5 & 6 \\ 7 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$,

$D = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$, $M = \begin{bmatrix} 15 & -9 \\ -5 & 3 \end{bmatrix}$, $N = \begin{bmatrix} 10 & 3 \\ 16/3 & 1 \end{bmatrix}$

$A\vec{b}_1 = \begin{bmatrix} 1\cdot 5 + 2\cdot 7 \\ 3\cdot 5 + 4\cdot 7 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$ $\quad A\vec{b}_2 = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$

$AB = \begin{bmatrix} 19 & 6 \\ 43 & 18 \end{bmatrix}$ $\quad BA = \begin{bmatrix} 23 & 34 \\ 7 & 14 \end{bmatrix} \neq AB$

$AI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$ $\quad IA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$

$AC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $\quad CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad (A^T)^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$ $\quad B^T = \begin{bmatrix} 5 & 7 \\ 6 & 0 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} 19 & 43 \\ 6 & 18 \end{bmatrix}$ $\quad B^T A^T = \begin{bmatrix} 19 & 43 \\ 6 & 18 \end{bmatrix} = (AB)^T$ $\quad A^T B^T = \begin{bmatrix} 23 & 7 \\ 34 & 14 \end{bmatrix} = (BA)^T$

$DM = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad DB = \begin{bmatrix} 26 & 6 \\ -52 & -12 \end{bmatrix}$ $\quad DN = \begin{bmatrix} 26 & 6 \\ -52 & -12 \end{bmatrix}$

Other observations:

$DM = 0$ (the 0 matrix) does not necessary mean that either D or M is the 0 matrix.

$DB = DN$ does not necessarily mean that $B = N$.

Why $CA = I$ is a very useful fact…

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$;and $\vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Note that $AC = I$.

Given $\quad A\vec{x} = \vec{b}$ $\qquad$ **So if** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$,

Then $\quad CA\vec{x} = C\vec{b}$

So $\quad I\vec{x} = C\vec{b}$

That is $\quad \vec{x} = C\vec{b}$ $\qquad$ **then** $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Preview of next section: what condition on a, b, c, d guarantees that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}$ has a unique solution, no matter what the right hand side $\begin{bmatrix} * \\ * \end{bmatrix}$ is?

$$\left[\begin{array}{cc|c} a & b & * \\ c & d & * \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & * \\ c & d & * \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & * \\ 0 & d - cb/a & * \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & * \\ 0 & 1 & * \end{array}\right]$$

In the final step we needed to be able to divide by $d - cb/a$ which is possible if $d - cb/a \neq 0$. So the condition that guarantees a unique solution is that $d - cb/a \neq 0$, i.e. $ad - bc \neq 0$.