Definitions and/or examples of words we've used in Chapter 1

Linear equation $2x_1 + x_2 - 4x_3 = 10$

Linear transformation $T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$.

For example, $T(x_1, x_2) = (x_1 + 3x_2, x_1 - x_2, 5x_2) = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Linear combination of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$.

Row reduction The process of putting a matrix into (reduced) echelon form.

Gaussian elimination One specific organized way of doing row reduction.

Pivot The leading "1" in each row when doing Gaussian Elimination.

Span (Noun) $span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is the set of all vectors that can be built (as a linear combination of) the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

Span (Verb) $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ spans R^m means everything in R^m can be built (as a linear combination of) the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

General solution All solutions of $A\vec{x} = \vec{b}$, usually of a form like $\vec{x}_p + c\vec{x}_h$ where $\vec{x}_p$ is a particular solution and where $\vec{x}_h$ is a homogenous solution.

Particular solution One solution (any solution!) of $A\vec{x} = \vec{b}$.

Homogenous problem The problem where the right hand side is $\vec{0}$, i.e. $A\vec{x} = \vec{0}$.

Homogenous solution The solution(s) $\vec{x}_h$ to the homogeneous problem $A\vec{x} = \vec{0}$.
If $\vec{v}_1, \dots, \vec{v}_n$ are all homogenous solutions, then so is any linear combination of them:

$$A(c_1\vec{v}_1 + \cdots c_n\vec{v}_n) = A(c_1\vec{v}_1) + \cdots + A(c_n\vec{v}_n) = c_1A\vec{v}_1 + \cdots + c_nA\vec{v}_n = c_1\vec{0} + \cdots + c_n\vec{0} = \vec{0}$$

Trivial solution The solution $\vec{x} = \vec{0}$ to $A\vec{x} = \vec{0}$.

Given matrix $A = [\vec{v}_1 \cdots \vec{v}_n]$:

Idea	Formal definition	Matrix version of formal definition	Intuition	One important consequence
Linear independence	$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$ only for $x_1 = \cdots = x_n = 0$	$A\vec{x} = \vec{0}$ only for $\vec{x} = \vec{0}$	None of the vectors $\vec{v}_i$ is a linear combination of the others.	For each RHS $\vec{b}$, there is at most one solution.
Linear dependence	$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$ for some $x_1, \dots, x_n$ not all 0	$A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$	One or more of the vectors $\vec{v}_i$ is a linear combination of the others.	If for a certain RHS $\vec{b}$ the problem $A\vec{x} = \vec{b}$ has a solution, then it has infinite solutions.

Three different ways to write the same problem:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 5x_1 + \quad -6x_3 &= 7 \end{aligned} \qquad x_1\begin{bmatrix}1\\5\end{bmatrix} + x_2\begin{bmatrix}2\\0\end{bmatrix} + x_3\begin{bmatrix}3\\-6\end{bmatrix} = \begin{bmatrix}4\\7\end{bmatrix} \qquad \begin{bmatrix}1 & 2 & 3\\5 & 0 & -6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\7\end{bmatrix}$$

All can be solved by row reducing the augmented matrix:

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 4\\5 & 0 & -6 & 7\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & 3 & 4\\0 & -10 & -21 & -13\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & 3 & 4\\0 & 1 & 2.1 & 1.3\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & -1.2 & 1.4\\0 & 1 & 2.1 & 1.3\end{array}\right]$$

Pivots in every row tell us there is a solution. The non-pivot columns tell us which variables are free variables. For this problem, we expected there to be infinite solutions because there were fewer equations than unknowns, that is, fewer rows than columns in the coefficient matrix.

From $\begin{aligned} x_1 \quad - 1.2x_3 &= 1.4 \\ x_2 + 2.1x_3 &= 1.3 \end{aligned}$ we get general solution of $\begin{aligned} x_1 &= 1.2x_3 + 1.4 \\ x_2 &= -2.1x_3 + 1.3 \\ x_3 &= \quad free \end{aligned}$, which can also be written as $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1.2x_3 + 1.4\\-2.1x_3 + 1.3\\x_3\end{bmatrix} = x_3\begin{bmatrix}1.2\\-2.1\\1\end{bmatrix} + \begin{bmatrix}1.4\\1.3\\0\end{bmatrix}$. Part of this general solution $\begin{bmatrix}1.2\\-2.1\\1\end{bmatrix}$ is the homogenous solution, that is, the solution to the homogenous problem:

$$\begin{bmatrix}1 & 2 & 3\\5 & 0 & -6\end{bmatrix}\begin{bmatrix}1.2\\-2.1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Any multiple of a homogenous solution is also a homogenous solution:

If $A\vec{x}_h = \vec{0}$, then $A(c\vec{x}_h) = cA\vec{x}_h = c\vec{0} = \vec{0}$.

In the general solution, $\begin{bmatrix}1.4\\1.3\\0\end{bmatrix}$ is one particular solution (of many). That is,

$$\begin{bmatrix}1 & 2 & 3\\5 & 0 & -6\end{bmatrix}\begin{bmatrix}1.4\\1.3\\0\end{bmatrix} = \begin{bmatrix}4\\7\end{bmatrix}.$$

Another particular solution (where x_3 is 1) is $\begin{bmatrix}2.6\\-0.8\\1\end{bmatrix}$, since $\begin{bmatrix}1 & 2 & 3\\5 & 0 & -6\end{bmatrix}\begin{bmatrix}2.6\\-0.8\\1\end{bmatrix} = \begin{bmatrix}4\\7\end{bmatrix}$.

In general, every solution $\vec{x}$ to the problem $A\vec{x} = \vec{b}$ is comprised of a particular solution $\vec{x}_p$, where $A\vec{x}_p = \vec{b}$, and a homogenous solution $\vec{x}_h$ (where $A\vec{x}_h = \vec{0}$): $\vec{x} = \vec{x}_p + c\vec{x}_h$.

We learn a lot about solutions to $A\vec{x} = \vec{b}$ by looking at just the coefficient matrix (i.e. the non-augmented matrix) $A = \begin{bmatrix}1 & 2 & 3\\5 & 0 & -6\end{bmatrix}$. Note that $\begin{bmatrix}1 & 2 & 3\\5 & 0 & -6\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1.2\\0 & 1 & 2.1\end{bmatrix}$. The two pivot columns tell us that these first two columns are the linearly independent columns of the three columns in A. (Bonus: we also see that $\begin{bmatrix}3\\-6\end{bmatrix} = -1.2\begin{bmatrix}1\\5\end{bmatrix} + 2.1\begin{bmatrix}2\\0\end{bmatrix}$.) Since these columns are linearly dependent, then *any* problem—in other words, the problem $A\vec{x} = \vec{b}$ for *any* $\vec{b}$—which has a solution will actually have an infinite number of solutions. (So in function talk, $T(\vec{x}) = Ax$ is not 1-to-1.) Next, $\begin{bmatrix}1 & 0 & -1.2\\0 & 1 & 2.1\end{bmatrix}$ has a pivot in every row—that is, there are no rows of all 0's—which tells us that $A\vec{x} = \vec{b}$ has a solution for every right hand side $\vec{b}$. (So in function talk, $T(\vec{x}) = Ax$ is onto.) So we can conclude that for $A = \begin{bmatrix}1 & 2 & 3\\5 & 0 & -6\end{bmatrix}$, the problem $A\vec{x} = \vec{b}$ has an infinite number of solutions no matter what $\vec{b}$ is. Of course what the solutions actually are will depend on what $\vec{b}$ is.