Most important ideas:

- There is not a lot new in this section, but we try to understand things a bit more deeply and we make some connections in this section.
- Every function $T(\vec{x})$ that is linear (see page 71) has a matrix A for which $T(\vec{x}) = A\vec{x}$.
- As we saw in section 1.8, we can think of matrix A "doing" something to vector $\vec{x}$, sometimes described as "A acting on $\vec{x}$", like we do functions: $T(\vec{x}) = A\vec{x}$ (the function T is a *linear transformation*, a special type of function).
- There is a geometric interpretation of what a 2×2 matrix A does to a vector $\vec{x}$ when you multiply $A\vec{x}$. See pages 72 – 75 for several examples.
- The function $T(\vec{x}) = A\vec{x}$ is sometimes referred to as a "mapping." That is, where A is of size $m \times n$, we "map" vector $\vec{x}$ from R^n to a vector $A\vec{x}$ in R^m. See the final page of this handout.
- Theorems 11 and 12 on pages 76 and 77 are important.
- Notation: vectors $\vec{e}_1, \vec{e}_2$, etc. are the columns of the identity matrix (see Book Example 1).

It turns out that any linear transformation $T(\vec{x})$ can be written with a matrix $T(\vec{x}) = A\vec{x}$. Given T, how do we find A, which is the standard matrix for the linear transformation T?

Suppose we are working with vectors in R^3. Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then we can write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3}$, and

$$T(\vec{x}) = \boldsymbol{T(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + x_3T(\vec{e}_3) = [T(\vec{e}_1)\ \ T(\vec{e}_2)\ \ T(\vec{e}_3)] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\vec{x}}.$$

This is Theorem 10 on page 71.

So where $T(\vec{x}) = A\vec{x}$, the ith column of A is simply $T(\vec{e}_i)$.

Problem 1: Suppose that for $T: R^3 \to R^2$ we have (with some randomly made up numbers)

$$T(x_1, x_2, x_3) = (\,2x_1 - 3x_2 + 4x_3\,, 5x_1 - 6x_3\,)\,.$$

Find A so that $T(\vec{x}) = A\vec{x}$. (Homework problems 1 and 2 are like this. Problem 3 and 7 are also similar, but a bit trickier.)

So I'm trying to find the matrix $\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$ **such that** $\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ 5x_1 \qquad\qquad - 6x_3 \end{bmatrix}$.

So I can see what the size of A is now, and I'll find the columns one at a time:

Column 1 is $T(\vec{e}_1) = T(1, 0, 0) = (2, 5)$.
Column 2 is $T(\vec{e}_2) = T(0, 1, 0) = (-3, 0)$.
Column 3 is $T(\vec{e}_3) = T(0, 0, 1) = (4, -6)$.

So $A = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 0 & -6 \end{bmatrix}$. Check: $\begin{bmatrix} 2 & -3 & 4 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ 5x_1 \qquad\qquad - 6x_3 \end{bmatrix}$.

It is also easy enough to find the standard matrix A by inspection. The fact that the ith column of A is $T(\vec{e}_i)$ is more useful as a *theoretical* result rather than to actually help us find A.

Problem 2: Find the standard matrix for $T: R^2 \to R^2$ where T rotates points (about the origin) through $-\pi/3$ radians, that is, 60° *clockwise*.

Rotating the point $(1,0)$ by $-60°$ is $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$. That is, $T(\vec{e}_1) = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$.

Rotating the point $(0,1)$ by $-60°$ is $(\frac{\sqrt{3}}{2}, \frac{1}{2})$. That is, $T(\vec{e}_2) = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$.

So the standard matrix is $[T(\vec{e}_1) \;\; T(\vec{e}_2)] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

Problem 3: Find the standard matrix for $T: R^2 \to R^2$ where T first rotates points through $-\pi/3$ radians, then reflects points through the y-axis.

Rotating $(1,0)$ by $-60°$ is $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$, then reflecting across the y-axis is $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, so $T(\vec{e}_1) = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$.

Rotating $(0,1)$ by $-60°$ is $(\frac{\sqrt{3}}{2}, \frac{1}{2})$, then reflecting across the y-axis is $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$, so $T(\vec{e}_2) = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$.

So the standard matrix is $[T(\vec{e}_1) \;\; T(\vec{e}_2)] = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

Or I could find the standard matrix for each part of the process:

$A_1 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ (found above in Problem 2) rotates by $-60°$.

$A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, since reflecting $(1,0)$ across the y-axis is $(-1,0)$ and reflecting $(0,1)$ across the y-axis is still $(0,1)$. Then the standard matrix for rotating then reflecting is

$$A_2 A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Why is it $A_2 A_1$ rather than $A_1 A_2$?

Because when a matrix does something to ("operates on") a vector, it is multiplying that vector on the left: $A_2(A_1\vec{v}) = A_2 A_1 \vec{v}$, so the transformation matrix is $A_2 A_1$.

See more examples in Book on pages 73 – 75.

Reminder of definition of linear independence: Suppose $A = [\vec{a}_1 \cdots \ \vec{a}_n]$ and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

The columns of A are linearly independent means $\boldsymbol{x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n = \vec{0}}$ **only if** $\boldsymbol{x_1 = \cdots = x_n = 0}$.

In other words, $\boldsymbol{A\vec{x} = \vec{0}}$ **only if** $\boldsymbol{\vec{x} = \vec{0}}$.

Problem 4: Consider $A\vec{x} = \vec{b}$. Suppose A is 2×3. What are the dimensions of $\vec{x}$ and $\vec{b}$?

$$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * \\ * \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix} \text{ so } \vec{x} \text{ is } 3 \times 1 \text{ and } \vec{b} \text{ is } 2 \times 1.$$

Suppose that $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

Then the linear transformation $T(\vec{x}) = A\vec{x}$ maps the vector $\vec{x} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ to the vector:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 8\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 9\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 76 \\ 100 \end{bmatrix}$$

So we see that the vector to which $\vec{x}$ is mapped by $T(\vec{x}) = A\vec{x}$ is simply the vector that is built out of the columns of A using the values of $\vec{x}$ as the weights for the columns.

Let's generalize from Problem 4 a bit. Suppose $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Remember that $\boldsymbol{A\vec{x} = \vec{b}}$ **means** $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$. That is, $\boldsymbol{\vec{b}}$ **can be built out of the columns of A,** where $\vec{x} = (x_1, x_2, x_3)$ tell us how to build $\vec{b}$ out of those columns.

Two questions we sometimes ask about (the columns of) matrix A:

Question 1: **Do the columns of A span R^2?** In "function talk," is linear transformation $T(\vec{x}) = A\vec{x}$ **onto**?

That is: **can every vector $\vec{b}$ in R^2 be built $\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$**?

That is: **for each $\vec{b}$ in R^2, is there an $\vec{x}$ such that $A\vec{x} = \vec{b}$**?

Question 2: **Are the columns of A linearly independent?** In "function talk," is the linear transformation $T(\vec{x}) = A\vec{x}$ **1-to-1**? See the following discussion.

Recall (from Section 1.7) that:

The **columns of matrix A are linearly independent** if and only if $\boldsymbol{A\vec{x} = \vec{0}}$ **only for** $\boldsymbol{\vec{x} = \vec{0}}$.

And since $T(\vec{x}) = A\vec{x}$, we can thus say:

The **columns of matrix A are linearly independent** if and only if $\boldsymbol{T(\vec{x}) = \vec{0}}$ **only for** $\boldsymbol{\vec{x} = \vec{0}}$.

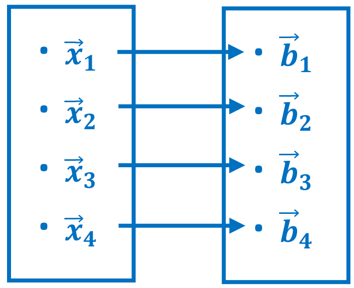
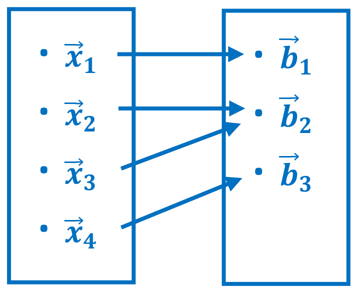
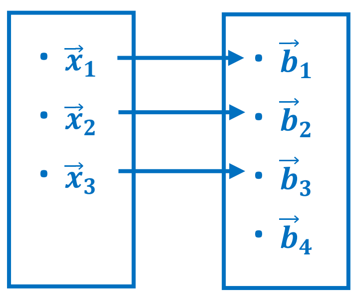
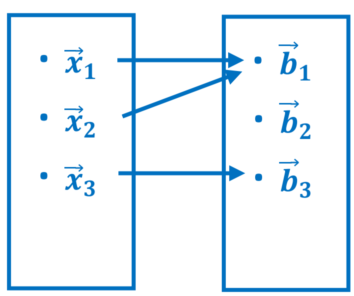
So by Theorem 11 in Section 1.9 we can conclude that:

The **columns of matrix A are linearly independent** if and only if function **T is one-to-one.**

The point of all of this is to describe in matrix terms (span, linearly independent) ideas that you are presumably already familiar with in function terms (onto, one-to-one).

Four possibilities for a matrix A.

Remember that $A\vec{x} = \vec{b}$ for some $\vec{x}$ means that $\vec{b}$ can be built out of (as a linear combination of) the columns of A. Compare this table to the table from Section 1.8.

Case	Picture	Example
Columns of A span R^m. Columns of A are linearly independent.		$A = \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$ $\sim \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ **Pivot in every row.** **Pivot in every column.**
Columns of A span R^m. Columns of A are not linearly independent.		$A = \begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$ $\sim \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\end{bmatrix}$ **Pivot in every row.** **Pivot not in every column.**
Columns of A do not span R^m. Columns of A are linearly independent.		$A = \begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix}$ $\sim \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$ **Pivot not in every row.** **Pivot in every column.**
Columns of A do not span R^m. Columns of A are not linearly independent.		$A = \begin{bmatrix}1 & -3\\ 2 & -6\end{bmatrix}$ $\sim \begin{bmatrix}1 & -3\\ 0 & 0\end{bmatrix}$ **Pivot not in every row.** **Pivot not in every column.**