Most important ideas:

- **First two pages of this handout: review of ideas from this class and ideas regarding functions.**
- **Definition of a linear transformation and more info, all on pages 65 – 66.**
- **Equations 4 and 5 on page 66 are simply Properties (i) and (ii) from page 65 combined.**

Review of some ideas regarding matrices and vectors

Let $m \times n$ matrix $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$ and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Recall that $A\vec{x} = \vec{b}$ means $\vec{b} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$,

so $\vec{b}$ is the vector we want to build, and $x_1, \ldots \ , x_n$ are the amounts of vectors $\vec{a}_1, \ldots, \vec{a}_n$ to build $\vec{b}$.

Recall: Suppose $\vec{a}_1, \ldots, \vec{a}_n$ are vectors from R^m. First, what do m and n mean?

$n =$ **number of vectors**

$m =$ **size of each vector**

For example, a 3×4 matrix has **4** vectors each of size **3** (i.e. from $\boldsymbol{R^3}$).

So we are trying to build vectors $\vec{b}$ of size **3** using the **4** vectors that are the columns of matrix A.

$\{\vec{a}_1, \ldots, \vec{a}_n\}$ spans R^m means **all vectors in $\boldsymbol{R^m}$ can be built using $\{\vec{a}_1, \ldots, \vec{a}_n\}$.** Good!

That is: **for each $\vec{b}$ in $\boldsymbol{R^m}$, there are weights $(x_1, \ldots, x_n)$ such that $\vec{b} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$.**

In other words: **for each $\vec{b}$ in $\boldsymbol{R^m}$, there is $\vec{x}$ such that $A\vec{x} = \vec{b}$.**

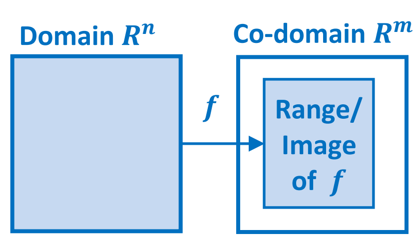
$\{\vec{a}_1, \ldots, \vec{a}_n\}$ does not span R^m means **there are vectors in $\boldsymbol{R^m}$ that can't be built using $\{\vec{a}_1, \ldots, \vec{a}_n\}$** . Bad!

That is: **there are vectors $\vec{b}$ in $\boldsymbol{R^m}$ for which there are no weights $(x_1, \ldots, x_n)$ such that $\vec{b} = x_1\vec{a}_1 + \cdots x_n\vec{a}_n$.**

In other words: **there are vectors $\vec{b}$ in $\boldsymbol{R^m}$ for which there is no $\vec{x}$ so that $A\vec{x} = \vec{b}$.**

Review of some ideas regarding functions

Consider function $f: R^n \to R^m$.
Sometimes we write $\vec{y} = f(\vec{x})$.



Also see Book Figure 2, p. 63

Domain: $\boldsymbol{R^n}$

Co-domain: $\boldsymbol{R^m}$

Image (or range) of f: **The part of $\boldsymbol{R^m}$ which f actually "maps" to.**

Example: Suppose that $f(x) = \sqrt{x-5} + 2$, so for this function we have $f : R \to R$.

This function f has domain of **real numbers $\geq$ 5** (since we can't have negative numbers in the $\sqrt{\ }$),
a co-domain of **all real numbers** and an image/range of **real numbers $\geq$ 2**.

What is the definition of 1-to-1?

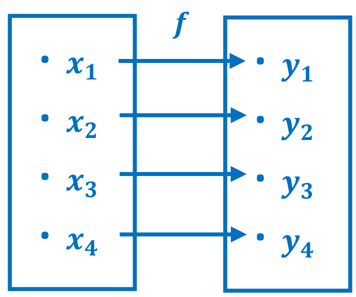
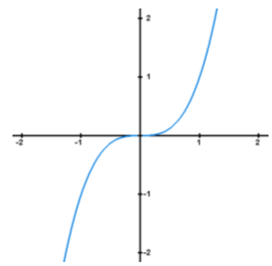
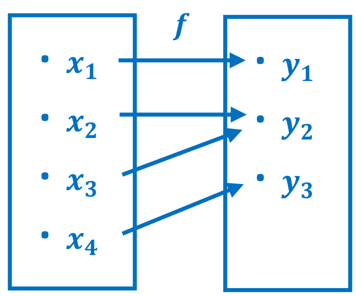
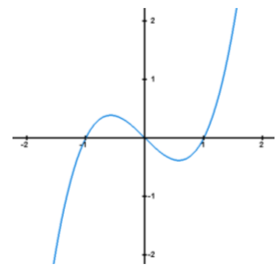
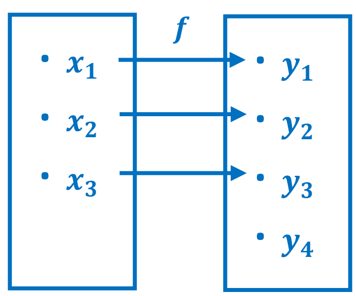
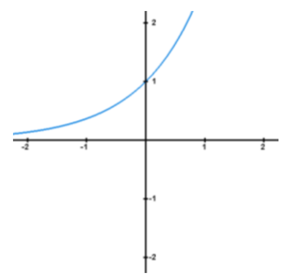
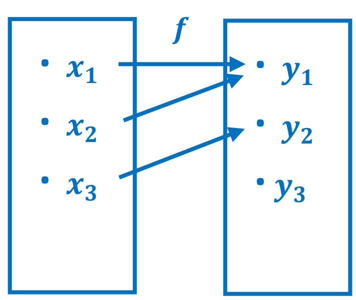
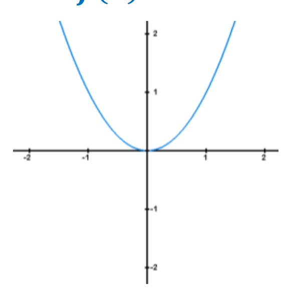
If $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, or equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

What is the definition of onto?

All elements in the co-domain are in the image.

Note: a function has an inverse only if it is **1-to-1** and it is **onto**. It's really good to have an inverse!

Four possibilities for a function $f(x)$.

Function	Diagram	Example
Onto 1-to-1		$f(x) = x^3$ 
Onto NOT 1-to-1		$f(x) = x^3 - x$ 
NOT Onto 1-to-1		$f(x) = e^x$ 
NOT Onto NOT 1-to-1		$f(x) = x^2$ 

Example 1: Consider the function $f(\vec{x}) = f(x_1, x_2) = (\,\boldsymbol{x_1 + x_2}\,,\ \boldsymbol{x_1 x_2}\,)$.
(This function represents any arbitrary function of two variables.)

Suppose $\vec{u} = \begin{bmatrix} \mathbf{8} \\ \mathbf{3} \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} \mathbf{2} \\ \mathbf{11} \end{bmatrix}$. (These represent two generic/random vectors in R^2.)

$f(\vec{u}) = \boldsymbol{f(8,3) = (8+3, 8\cdot 3) = (11, 24).}$

$f(\vec{v}) = \boldsymbol{f(2,11) = (13, 22).}$

$f(2\vec{u} + 3\vec{v}) = \boldsymbol{f(22,39) = (61, 858).}$

Is $f(2\vec{u} + 3\vec{v}) = 2f(\vec{u}) + 3f(\vec{v})$? **No:** $\boldsymbol{(61, 858) \neq 2(11, 24) + 3(13, 22).}$

In fact, for most functions, $f(2\vec{u} + 3\vec{v}) \neq 2f(\vec{u}) + 3f(\vec{v})$, that is, they are not "linear" (see definition of linear in book on page 65).

Example 2: Where $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, define $f(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1 + 2x_2, -3x_1 + 4x_2)$.

Suppose $\vec{u} = \begin{bmatrix} \mathbf{8} \\ \mathbf{3} \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} \mathbf{2} \\ \mathbf{11} \end{bmatrix}$, as in Example 1.

$f(\vec{u}) = \boldsymbol{(14, -12).}$

$f(\vec{v}) = \boldsymbol{(24, 38).}$

$f(2\vec{u} + 3\vec{v}) = \boldsymbol{f(22,39) = (100, 90).}$

Is $f(2\vec{u} + 3\vec{v}) = 2f(\vec{u}) + 3f(\vec{v})$? **Yes:** $\boldsymbol{(100, 90) = 2(14, -12) + 3(24, 38).}$

A linear transformation/function is one for which $f(c_1\vec{u} + c_2\vec{v}) = c_1 f(\vec{u}) + c_2 f(\vec{v})$.
See Book Example 1, p. 64.

See Book Example 3 about a shear transformation, p. 65 (The author is making a bit of a joke with the image next to Figure 4.)

See Book Examples 5 and 6, p. 67.