

Most important ideas:

- **First two pages of this handout:** review of ideas from this class and ideas regarding functions.
- **Definition of a linear transformation** and more info, all on pages 65 – 66.
- **Equations 4 and 5 on page 66** are simply Properties (i) and (ii) from page 65 combined.

Review of some ideas regarding matrices and vectors

Let $m \times n$ matrix $A = [\vec{a}_1 \cdots \vec{a}_n]$ and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Recall that $A\vec{x} = \vec{b}$ means $\vec{b} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$, so \vec{b} is the vector we want to build, and x_1, \dots, x_n are the amounts of vectors $\vec{a}_1, \dots, \vec{a}_n$ to build \vec{b} .

Recall: Suppose $\vec{a}_1, \dots, \vec{a}_n$ are vectors from R^m . First, what do m and n mean?

$n =$ **number of vectors**

$m =$ **size of each vector**

For example, a 3×4 matrix has **4** vectors each of size **3** (i.e. from R^3).

So we are trying to build vectors \vec{b} of size **3** using the **4** vectors that are the columns of matrix A .

$\{\vec{a}_1, \dots, \vec{a}_n\}$ spans R^m means **all vectors in R^m can be built using $\{\vec{a}_1, \dots, \vec{a}_n\}$** . Good!

That is: **for each \vec{b} in R^m , there are weights (x_1, \dots, x_n) such that $\vec{b} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$** .

In other words: **for each \vec{b} in R^m , there is \vec{x} such that $A\vec{x} = \vec{b}$** .

$\{\vec{a}_1, \dots, \vec{a}_n\}$ does not span R^m means **there are vectors in R^m that can't be built using $\{\vec{a}_1, \dots, \vec{a}_n\}$** . Bad!

That is: **there are vectors \vec{b} in R^m for which there are no weights (x_1, \dots, x_n) such that $\vec{b} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$** .

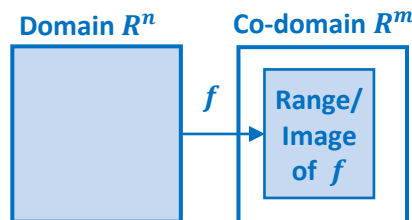
In other words: **there are vectors \vec{b} in R^m for which there is no \vec{x} so that $A\vec{x} = \vec{b}$** .

Review of some ideas regarding functions

Consider function $f: R^n \rightarrow R^m$. Sometimes we write $\vec{y} = f(\vec{x})$.

Domain: R^n

Co-domain: R^m



Also see Book Figure 2, p. 63

Image (or range) of f : **The part of R^m which f actually "maps" to.**

Example: Suppose that $f(x) = \sqrt{x-5} + 2$, so for this function we have $f: R \rightarrow R$.

This function f has domain of **real numbers ≥ 5** (since we can't have negative numbers in the $\sqrt{\quad}$), a co-domain of **all real numbers** and an image/range of **real numbers ≥ 2** .

What is the definition of 1-to-1?

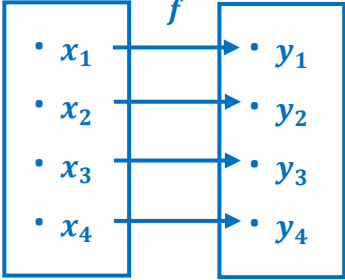
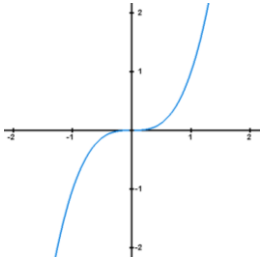
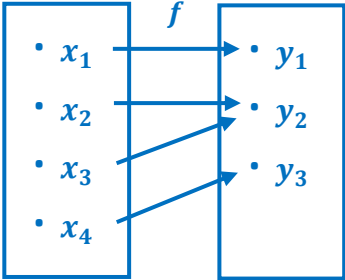
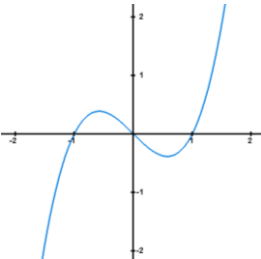
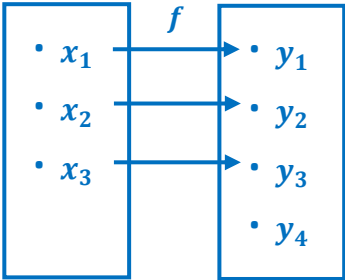
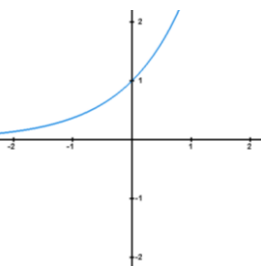
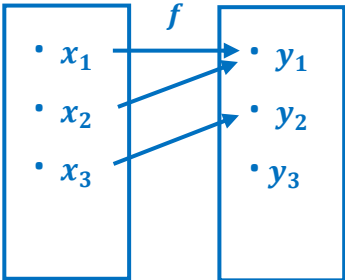
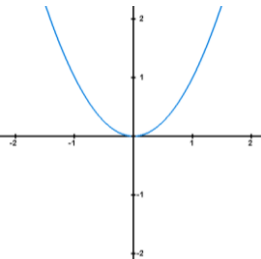
If $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, or equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

What is the definition of onto?

All elements in the co-domain are in the image.

Note: a function has an inverse only if it is **1-to-1** and it is **onto**. It's really good to have an inverse!

Four possibilities for a function $f(x)$.

Function	Diagram	Example
Onto 1-to-1	 <p>A mapping diagram with a left box containing x_1, x_2, x_3, x_4 and a right box containing y_1, y_2, y_3, y_4. Arrows labeled f point from each x_i to its corresponding y_i.</p>	$f(x) = x^3$  <p>A Cartesian coordinate system showing the graph of $f(x) = x^3$. The x-axis and y-axis both range from -2 to 2. The curve passes through the origin and is strictly increasing.</p>
Onto NOT 1-to-1	 <p>A mapping diagram with a left box containing x_1, x_2, x_3, x_4 and a right box containing y_1, y_2, y_3, y_4. Arrows labeled f point from x_1 to y_1, x_2 to y_2, and both x_3 and x_4 to y_3. y_4 has no arrow pointing to it.</p>	$f(x) = x^3 - x$  <p>A Cartesian coordinate system showing the graph of $f(x) = x^3 - x$. The x-axis and y-axis both range from -2 to 2. The curve is a cubic with a local maximum and a local minimum, passing through the origin.</p>
NOT Onto 1-to-1	 <p>A mapping diagram with a left box containing x_1, x_2, x_3 and a right box containing y_1, y_2, y_3, y_4. Arrows labeled f point from x_1 to y_1, x_2 to y_2, and x_3 to y_3. y_4 has no arrow pointing to it.</p>	$f(x) = e^x$  <p>A Cartesian coordinate system showing the graph of $f(x) = e^x$. The x-axis ranges from -2 to 2 and the y-axis from -2 to 2. The curve is strictly increasing and concave up, passing through the point (0, 1).</p>
NOT Onto NOT 1-to-1	 <p>A mapping diagram with a left box containing x_1, x_2, x_3 and a right box containing y_1, y_2, y_3. Arrows labeled f point from x_1 and x_2 to y_1, and from x_3 to y_2. y_3 has no arrow pointing to it.</p>	$f(x) = x^2$  <p>A Cartesian coordinate system showing the graph of $f(x) = x^2$. The x-axis and y-axis both range from -2 to 2. The curve is a parabola opening upwards with its vertex at the origin.</p>

Example 1: Consider the function $f(\vec{x}) = f(x_1, x_2) = (x_1 + x_2, x_1x_2)$.

(This function represents any arbitrary function of two variables.)

Suppose $\vec{u} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$. (These represent two generic/random vectors in R^2 .)

$$f(\vec{u}) = f(8, 3) = (8 + 3, 8 \cdot 3) = (11, 24).$$

$$f(\vec{v}) = f(2, 11) = (13, 22).$$

$$f(2\vec{u} + 3\vec{v}) = f(22, 39) = (61, 858).$$

Is $f(2\vec{u} + 3\vec{v}) = 2f(\vec{u}) + 3f(\vec{v})$? **No:** $(61, 858) \neq 2(11, 24) + 3(13, 22)$.

In fact, for most functions, $f(2\vec{u} + 3\vec{v}) \neq 2f(\vec{u}) + 3f(\vec{v})$, that is, they are not “linear” (see definition of linear in book on page 65).

Example 2: Where $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, define $f(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1 + 2x_2, -3x_1 + 4x_2)$.

Suppose $\vec{u} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$, as in Example 1.

$$f(\vec{u}) = (14, -12).$$

$$f(\vec{v}) = (24, 38).$$

$$f(2\vec{u} + 3\vec{v}) = f(22, 39) = (100, 90).$$

Is $f(2\vec{u} + 3\vec{v}) = 2f(\vec{u}) + 3f(\vec{v})$? **Yes:** $(100, 90) = 2(14, -12) + 3(24, 38)$.

A linear transformation/function is one for which $f(c_1\vec{u} + c_2\vec{v}) = c_1f(\vec{u}) + c_2f(\vec{v})$.

See Book Example 1, p. 64.

See Book Example 3 about a shear transformation, p. 65 (The author is making a bit of a joke with the image next to Figure 4.)

See Book Examples 5 and 6, p. 67.