Math 260 Section 1.7

Human subtlety will never devise an invention more beautiful, more simple or more direct than does nature because in her inventions nothing is lacking, and nothing is superfluous. —Leonardo da Vinci

Most important ideas:

- **Given a set of vectors, there are two things we really want:**
  - **To be able to build any vector out of a given set of vectors (of the same size).**
  - **To do it as efficiently as possible—today's topic.**
- ***Definition* of <u>linear dependence</u> and <u>linear independence</u>, page 56.**
- **The important idea in box on page 57.**
- ***Intuition* of <u>linear independence</u>: in a given set of vectors, none is redundant or unnecessary.** ***Intuition* of <u>linear dependence</u>: we could build the same vectors even if we got rid of one or more of the set, i.e. one or more of the vectors is redundant or unnecessary.**
- **If the number of vectors in a given set of vectors is greater than the size of each vector (for example, 3 vectors from R^2), then the vectors are linearly dependent. Any set of vectors containing the zero vector is linearly dependent.**

First idea: Unneeded vectors or not? We want to build a given vector (create a linear combination) using $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ and/or $\begin{bmatrix}5\\6\end{bmatrix}$. We'll "randomly" choose the vector $\begin{bmatrix}-1\\1\end{bmatrix}$ to represent all vectors in R^2.

Problem 1: Can we build $\begin{bmatrix}-1\\1\end{bmatrix}$ using only $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$? **Yes. Let's find the values of c_1 and c_2 so that $c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$: $\left[\begin{array}{cc|c}1&3&-1\\2&4&1\end{array}\right] \sim \left[\begin{array}{cc|c}1&0&7/2\\0&1&-3/2\end{array}\right]$, so $\frac{7}{2}\begin{bmatrix}1\\2\end{bmatrix}-\frac{3}{2}\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$.**

We could also build $\begin{bmatrix}-1\\1\end{bmatrix}$ using only $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}5\\6\end{bmatrix}$: $\frac{11}{4}\begin{bmatrix}1\\2\end{bmatrix}-\frac{3}{4}\begin{bmatrix}5\\6\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$

or using only $\begin{bmatrix}3\\4\end{bmatrix}$ and $\begin{bmatrix}5\\6\end{bmatrix}$: $\frac{11}{2}\begin{bmatrix}3\\4\end{bmatrix}-\frac{7}{2}\begin{bmatrix}5\\6\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$

So we can build $\begin{bmatrix}-1\\1\end{bmatrix}$ using $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ and $\begin{bmatrix}5\\6\end{bmatrix}$ in different ways. We don't need all three of $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ and $\begin{bmatrix}5\\6\end{bmatrix}$. We can get rid of one of them and still be able to build the same vectors that we could build with all three of them.

Now notice: $\frac{7}{2}\begin{bmatrix}1\\2\end{bmatrix}-\frac{3}{2}\begin{bmatrix}3\\4\end{bmatrix}=\frac{11}{4}\begin{bmatrix}1\\2\end{bmatrix}-\frac{3}{4}\begin{bmatrix}5\\6\end{bmatrix} \Rightarrow \begin{bmatrix}1\\2\end{bmatrix}=2\begin{bmatrix}3\\4\end{bmatrix}-\begin{bmatrix}5\\6\end{bmatrix}$

which means $\begin{bmatrix}3\\4\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix}+\frac{1}{2}\begin{bmatrix}5\\6\end{bmatrix}$ and $\begin{bmatrix}5\\6\end{bmatrix}=-\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}3\\4\end{bmatrix}$

and also that $\begin{bmatrix}1\\2\end{bmatrix}-2\begin{bmatrix}3\\4\end{bmatrix}+\begin{bmatrix}5\\6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ that is, $\begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

In general, given some vectors, the following four statements are equivalent (if any one statement is true, then so are the other three):

- **We have extra, unneeded vectors.**
- **One or more vectors can be built as a linear combination of the other vectors.**
- **There is a non-trivial way to build the zero vector $\vec{0}$ from the given vectors.**
- **Where the columns of A are the given vectors, $A\vec{x}=\vec{0}$ for some $\vec{x}\neq\vec{0}$ ("non-trivial" $\vec{x}$)**

Problem 2a: For what values of x_1, x_2 and x_3 is $\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$?

Problem 2b: For what values of x_1, x_2 and x_3 is $x_1\begin{bmatrix}1\\ 2\end{bmatrix} + x_2\begin{bmatrix}3\\ 4\end{bmatrix} + x_3\begin{bmatrix}5\\ 6\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$?

$$\left[\begin{array}{ccc|c}\mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{0}\\ \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathbf{0}\end{array}\right] \sim \left[\begin{array}{ccc|c}\mathbf{1} & \mathbf{0} & -\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{0}\end{array}\right] \textbf{ so } \begin{bmatrix}\boldsymbol{x_1}\\ \boldsymbol{x_2}\\ \boldsymbol{x_3}\end{bmatrix} = \begin{bmatrix}\boldsymbol{x_3}\\ -\mathbf{2}\boldsymbol{x_3}\\ \boldsymbol{x_3}\end{bmatrix} = \boldsymbol{x_3}\begin{bmatrix}\mathbf{1}\\ -\mathbf{2}\\ \mathbf{1}\end{bmatrix}.$$

So we see that the set of vectors $\left\{\begin{bmatrix}1\\ 2\end{bmatrix}, \begin{bmatrix}3\\ 4\end{bmatrix}, \begin{bmatrix}5\\ 6\end{bmatrix}\right\}$ is linearly dependent.

A few specific values are: $\quad (\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}) = (\mathbf{1}, -\mathbf{2}, \mathbf{1})$

or $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}) = (-\mathbf{3},\ \mathbf{6}, -\mathbf{3})$

or $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}) = (\mathbf{1/4}, -\mathbf{1/2}, \mathbf{1/4})$

Using the solution $(x_1, x_2, x_3) = (1, -2, 1)$ we can see that:

$\begin{bmatrix}\mathbf{1}\\ \mathbf{2}\end{bmatrix} = \mathbf{2}\begin{bmatrix}\mathbf{3}\\ \mathbf{4}\end{bmatrix} - \mathbf{1}\begin{bmatrix}\mathbf{5}\\ \mathbf{6}\end{bmatrix}$ **or equivalently** $\begin{bmatrix}\mathbf{3}\\ \mathbf{4}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\mathbf{1}\\ \mathbf{2}\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\mathbf{5}\\ \mathbf{6}\end{bmatrix}$ **or equivalently** $\begin{bmatrix}\mathbf{5}\\ \mathbf{6}\end{bmatrix} = -\mathbf{1}\begin{bmatrix}\mathbf{1}\\ \mathbf{2}\end{bmatrix} + \mathbf{2}\begin{bmatrix}\mathbf{3}\\ \mathbf{4}\end{bmatrix}$.

Linear dependence means that one (or more) of the vectors is a linear combination of the other vectors. Linear <u>in</u>dependence means just the opposite: none of the vectors is a linear combination of the others. See Theorem 7 on page 58. This is not the formal definition—that definition is on page 56—but it is some important and useful intuition. It turns out that the pivot columns indicate which of the original vectors are non-redundant (the vectors that are redundant could be eliminated from the set and you could still build the same vectors).

Problem 3: Is the set $\left\{\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}, \begin{bmatrix}4\\ 5\\ 6\end{bmatrix}, \begin{bmatrix}7\\ 8\\ 9\end{bmatrix}\right\}$ linearly independent? **No:** $\left[\begin{array}{ccc|c}\mathbf{1} & \mathbf{4} & \mathbf{7} & \mathbf{0}\\ \mathbf{2} & \mathbf{5} & \mathbf{8} & \mathbf{0}\\ \mathbf{3} & \mathbf{6} & \mathbf{9} & \mathbf{0}\end{array}\right] \sim \left[\begin{array}{ccc|c}\mathbf{1} & \mathbf{0} & -\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]$.

Notice that $\begin{bmatrix}\mathbf{1} & \mathbf{4} & \mathbf{7}\\ \mathbf{2} & \mathbf{5} & \mathbf{8}\\ \mathbf{3} & \mathbf{6} & \mathbf{9}\end{bmatrix}\begin{bmatrix}\mathbf{1}\\ -\mathbf{2}\\ \mathbf{1}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}$, **that is,** $\mathbf{1}\begin{bmatrix}\mathbf{1}\\ \mathbf{2}\\ \mathbf{3}\end{bmatrix} + (-\mathbf{2})\begin{bmatrix}\mathbf{4}\\ \mathbf{5}\\ \mathbf{6}\end{bmatrix} + \mathbf{1}\begin{bmatrix}\mathbf{7}\\ \mathbf{8}\\ \mathbf{9}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}$.

(or any multiple of ↑)

The first two columns are pivot columns, and the third is not a pivot column, which means that it is redundant, it is not needed. That is, we can build any vector out of the first two that we could out of all three vectors. That is, $\boldsymbol{span}\left\{\begin{bmatrix}\mathbf{1}\\ \mathbf{2}\\ \mathbf{3}\end{bmatrix}, \begin{bmatrix}\mathbf{4}\\ \mathbf{5}\\ \mathbf{6}\end{bmatrix}\right\} = \boldsymbol{span}\left\{\begin{bmatrix}\mathbf{1}\\ \mathbf{2}\\ \mathbf{3}\end{bmatrix}, \begin{bmatrix}\mathbf{4}\\ \mathbf{5}\\ \mathbf{6}\end{bmatrix}, \begin{bmatrix}\mathbf{7}\\ \mathbf{8}\\ \mathbf{9}\end{bmatrix}\right\}$.

Also notice that the third vector is $-\mathbf{1}$ times the first vector plus 2 times the second vector. We can actually see this information in the third column.

Problem 4: Is the set $\left\{\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}, \begin{bmatrix}4\\ 8\\ 12\end{bmatrix}, \begin{bmatrix}7\\ 8\\ 9\end{bmatrix}\right\}$ linearly independent? $\begin{bmatrix}\mathbf{1} & \mathbf{4} & \mathbf{7}\\ \mathbf{2} & \mathbf{5} & \mathbf{8}\\ \mathbf{3} & \mathbf{6} & \mathbf{9}\end{bmatrix} \sim \cdots \sim \begin{bmatrix}\mathbf{1} & \mathbf{4} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}$.

So we see that the second column is redundant, that is, $\boldsymbol{span}\left\{\begin{bmatrix}\mathbf{1}\\ \mathbf{2}\\ \mathbf{3}\end{bmatrix}, \begin{bmatrix}\mathbf{7}\\ \mathbf{8}\\ \mathbf{9}\end{bmatrix}\right\} = \boldsymbol{span}\left\{\begin{bmatrix}\mathbf{1}\\ \mathbf{2}\\ \mathbf{3}\end{bmatrix}, \begin{bmatrix}\mathbf{4}\\ \mathbf{8}\\ \mathbf{12}\end{bmatrix}, \begin{bmatrix}\mathbf{7}\\ \mathbf{8}\\ \mathbf{9}\end{bmatrix}\right\}$.

We also see that the second vector is 4 times the first vector (which of course was obvious).

Final thought: make sure you understand what is being shown in Figure 2 on page 58.