Most important ideas:

- **The homogeneous problem $A\vec{x} = \vec{0}$:**
  - **The equation $A\vec{x} = \vec{0}$ always has (at least) one solution: $\vec{x} = \vec{0}$**
  - **The equation $A\vec{x} = \vec{0}$ has a non-trivial (non-zero) solution if and only if the equation has at least one free variable.**
  - **Any multiple of a homogeneous solution is also a homogeneous solution.**
- **Each solution to the non-homogenous problem consists of a particular solution $\vec{x}_p$ (where $A\vec{x}_p = \vec{b}$) and a homogenous solution $\vec{x}_h$ (where $A\vec{x}_h = \vec{0}$). My notation here is a bit different from the book's.**
- **See steps on page 46 for coming up with a solution set to $A\vec{x} = \vec{b}$.**

Problem 1 (compare to class Problem 4 from Section 1.4): What is the solution to $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

$$\left[\begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right] \Rightarrow \begin{matrix} x_1 & & -x_3 = 0 \\ & x_2 & +2x_3 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = x_3 \\ x_2 = -2x_3 \\ x_3 = free \end{matrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

There is a pivot in every row, which means **there will be a solution.**

There is an extra column $\begin{matrix} -1 \\ 2 \end{matrix}$ without a pivot, which tells us **there will be infinite solutions: the third column has no pivot and thus the third unknown will be a free variable.**

Problem 2 (compare this problem to Problem 1): What is the solution to $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$?

$$\left[\begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{matrix}\middle|\begin{matrix} 1 \\ 6 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{matrix}\middle|\begin{matrix} 7 \\ -2 \end{matrix}\right] \Rightarrow \begin{matrix} x_1 & & -x_3 = 7 \\ & x_2 & +2x_3 = -2 \end{matrix} \Rightarrow \begin{matrix} x_1 = 7 + x_3 \\ x_2 = -2 - 2x_3 \\ x_3 = free \end{matrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 + x_3 \\ -2 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

This is the general solution. There are infinite solutions (since there is a free variable), so we can't list all of them. What we can say/write is that every solution can be written as $\begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ for any constant c. **Examples: For $c = 0$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$. (Check that $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.) For $c = 1$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$. And so on.**

Compare this general solution for Problem 2 to the solution we found for Problem 1:

$\begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$ **is a *particular* solution and** $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ **is the *homogeneous* solution, that is, the solution to *homogeneous* problem, the problem for which the right hand side is $\vec{0}$.**

How we write the general solution is not unique. Alternate forms of general solution:

$$\begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{or} \ \ldots$$

Problem 3: Consider $\begin{aligned} x_1 + \ \ 2x_2 + 4x_3 + 5x_4 + 5x_5 &= \ \ 2 \\ 3x_1 + 6x_2 + 8x_3 + 7x_4 + 3x_5 &= 10 \end{aligned}$ with augmented matrix $\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 5 & 2 \\ 3 & 6 & 8 & 7 & 3 & 10 \end{array}\right]$.

How many solutions do you expect this problem to have (0 or 1 or ∞)? ∞

Why? **# equations < # unknowns**.

How many pivots do you predict there will be? **2, since there are 2 rows.**

How many non-pivot columns then do you expect there will be?
3, since there are 5 columns and just 2 rows, so 3 more columns than there are rows.

How many free variables do you think there will be? **3, the number of expected non-pivot columns.**

Let's find the solution:

$$\left[\begin{array}{ccccc|c} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{5} & \mathbf{5} & \mathbf{2} \\ \mathbf{3} & \mathbf{6} & \mathbf{8} & \mathbf{7} & \mathbf{3} & \mathbf{10} \end{array}\right] \sim \left[\begin{array}{ccccc|c} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{5} & \mathbf{5} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{-4} & \mathbf{-8} & \mathbf{-12} & \mathbf{4} \end{array}\right] \sim \left[\begin{array}{ccccc|c} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{-3} & \mathbf{-7} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{-1} \end{array}\right]\text{, so}$$

$$\begin{aligned} \boldsymbol{x_1 + 2x_2 \quad - 3x_4 - 7x_5} &= \mathbf{6} \\ \boldsymbol{x_3 + 2x_4 + 3x_5} &= \mathbf{-1} \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 + 3x_4 + 7x_5 \\ x_2 \\ -1 - 2x_4 - 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Notice the two parts of the solution:

the particular solution $\begin{bmatrix} 6 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ **and the homogenous solutions** $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$**, one for each free variable, corresponding to the non-pivot columns. Plug each one into the original two equations above to see what you get.**

<u>True</u>/False: If $\vec{x}_h$ is a solution to $A\vec{x} = \vec{0}$, then so is any multiple $c\vec{x}_h$ (for any scalar c) of $\vec{x}_h$.

If $A\vec{x}_h = \vec{0}$, then $A(c\vec{x}_h) = c\,A\vec{x}_h = c\vec{0} = \vec{0}$. Easy to prove!

For example, since $\begin{bmatrix} 1 & 2 & 4 & 5 & 5 \\ 3 & 6 & 8 & 7 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\begin{bmatrix} 1 & 2 & 4 & 5 & 5 \\ 3 & 6 & 8 & 7 & 3 \end{bmatrix} \begin{bmatrix} -15 \\ 0 \\ 10 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We will discuss problem 1.5.25 for a minute next time. Try to understand it on your own before coming to class.