Most important ideas:

- **Matrix-vector product, on page 35.**
- **Solution to $A\vec{x} = \vec{b}$, etc., Theorem 3, on page 36.**
- **Equivalence of four statements in Theorem 4, page 37, in particular item (d) about A having a pivot position in every row.**

Problem 1 (compare to class Problems 3a, 5a and 5b from Section 1.3):
Given any vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in R^2, are there x_1 and x_2 so that $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \vec{b}$?
That is, is there a solution x_1 and x_2 to $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, no matter what $\vec{b}$ is?

Yes. $\left[\begin{array}{cc|c} \mathbf{1} & \mathbf{3} & \boldsymbol{b_1} \\ \mathbf{2} & \mathbf{1} & \boldsymbol{b_2} \end{array}\right] \sim \left[\begin{array}{cc|c} \mathbf{1} & \mathbf{0} & -\frac{1}{5}\boldsymbol{b_1} + \frac{3}{5}\boldsymbol{b_2} \\ \mathbf{0} & \mathbf{1} & \frac{2}{5}\boldsymbol{b_1} - \frac{1}{5}\boldsymbol{b_2} \end{array}\right]$, **so** $\begin{array}{l} \boldsymbol{x_1} = -\frac{1}{5}\boldsymbol{b_1} + \frac{3}{5}\boldsymbol{b_2} \\ \boldsymbol{x_2} = \ \ \frac{2}{5}\boldsymbol{b_1} - \frac{1}{5}\boldsymbol{b_2} \end{array}$.

Of course we see that the particular values of x_1 and x_2 depend on the values of b_1 and b_2.

Is there a pivot in every row of the coefficient matrix?

Yes. So there is guaranteed to be a solution.

Problem 2 (compare to class Problem 6 from Section 1.3): Given any right hand side $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, is there a solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to $\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$?

No. $\left[\begin{array}{cc|c} \mathbf{1} & \mathbf{-3} & \boldsymbol{b_1} \\ \mathbf{2} & \mathbf{-6} & \boldsymbol{b_2} \end{array}\right] \sim \left[\begin{array}{cc|c} \mathbf{1} & \mathbf{-3} & \boldsymbol{b_1} \\ \mathbf{0} & \mathbf{0} & -\mathbf{2}\boldsymbol{b_1} + \boldsymbol{b_2} \end{array}\right]$, **so in general there is no solution (there is only a solution if $-2b_1 + b_2 = 0$, that is, if $b_2 = 2b_1$).**

Is there a pivot in every row of the coefficient matrix?

No. So there may not be a solution, depending on what the values of b_1 and b_2 are.

What restriction on b_1 and b_2 is there?

That $-2b_1 + b_2 = 0$, that is, that $b_2 = 2b_1$, that is, $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 2b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so there is a solution to $\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ only if $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is some multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Note also that if there is a solution, there will be an infinite number of solutions, as the second variable will be a free variable. We can tell this since the second column in the coefficient matrix is not a pivot column.

Problem 3 (compare to class Problems 7 and 8 from Section 1.3): Given any right hand side $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, is there a solution to $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$? That is, do $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ span R^3?

$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 5 & b_2 \\ 3 & 6 & b_3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & -3 & -2b_1 + b_2 \\ 0 & 0 & b_1 - 2b_2 + b_3 \end{array}\right]$, **so there is no solution if** $b_1 - 2b_2 + b_3 \neq 0$.

Is there a pivot in every row of the coefficient matrix?

No. So there may not be a solution, depending on the values of b_1, b_2 **and** b_3.

For there to be a solution, what restrictions on b_1, b_2 and b_3 are there?

That $b_1 - 2b_2 + b_3 = 0$. **Note that this is the equation for a plane.**

Problem 4 (compare to Problems 9 and 10 from Section 1.3): Given any right hand side $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, is there a solution to $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$? That is, do $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$ span R^2?

$\left[\begin{array}{ccc|c} 1 & 3 & 5 & b_1 \\ 2 & 4 & 6 & b_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & b_1 \\ 0 & -2 & -4 & -2b_1 + b_2 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2b_1 + \frac{3}{2}b_2 \\ 0 & 1 & 2 & b_1 - \frac{1}{2}b_2 \end{array}\right]$

Is there a pivot in every row of the coefficient matrix?

Yes. So there will be a solution regardless of the values of b_1 **and** b_2.

Are there any restrictions on b_1 and b_2?

Nope. They can be any value and there will still be a solution.

How many solutions are there? (Notice Column 3 has no pivot. It's sort of an "extra" column.)

Infinitely many. The third column with no pivot means there will be a free variable x_3 **so that there will be an infinite number of solutions.**

For example, if $b_1 = 1$ **and** $b_2 = 6$, **then** $\begin{aligned} -2b_1 + \tfrac{3}{2}b_2 &= 7 \\ b_1 - \tfrac{1}{2}b_2 &= -2 \end{aligned}$ **so our row reduced augmented matrix is** $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & 1 & 2 & -2 \end{array}\right]$, **so** $\begin{aligned} x_1 &= 7 + x_3 \\ x_2 &= -2 - 2x_3 \\ x_3 &= x_3 \end{aligned}$. **This is called the <u>general solution</u>.**

A couple of <u>particular solutions</u> are $\begin{aligned} x_1 &= 7 \\ x_2 &= -2 \\ x_3 &= 0 \end{aligned}$ **and** $\begin{aligned} x_1 &= 8 \\ x_2 &= -4 \\ x_3 &= 1 \end{aligned}$. **(There are infinitely many.)**

Check: $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. ✓

$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. ✓