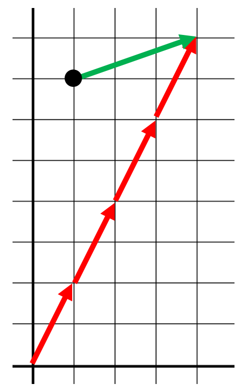
Use of technology in row reduction: links on are class homepage.

Most important ideas:

- **Linear combination, on page 27.**
- **Connection between vector equation and linear system, on page 29.**
- **Span, on page 30.**

Problem 1:
Can the point (1,7) be reached by using the directions of **(1,2)** and **(3,1)**?

Yes. Move in the direction (1,2) forwards four times then in the direction (3,1) backwards one time, as shown in the diagram at right.



Problem 2: Is $\begin{bmatrix}1\\7\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$?

Yes. There are weights/constants c_1 and c_2 such that $\begin{bmatrix}1\\7\end{bmatrix} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\1\end{bmatrix}$. See work below.

Problem 3a: Are there c_1 and c_2 so that $c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}1\\7\end{bmatrix}$?

Yes. We can rewrite the problem as trying to find c_1 and c_2 such that $\begin{bmatrix}c_1\\2c_1\end{bmatrix} + \begin{bmatrix}3c_2\\c_2\end{bmatrix} = \begin{bmatrix}1\\7\end{bmatrix}$, that is, $\begin{matrix}c_1 + 3c_2 = 1\\2c_1 + c_2 = 7\end{matrix}$, which has solution of $c_1 = 4, c_2 = -1$, as shown below.

Problem 3b: Are there c_1 and c_2 so that $\begin{matrix}c_1 + 3c_2 = 1\\2c_1 + c_2 = 7\end{matrix}$?

Yes. The augmented matrix and row reduction (Gaussian Elimination):

$$\left[\begin{array}{cc|c}1 & 3 & 1\\2 & 1 & 7\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & 1\\0 & -5 & 5\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & 1\\0 & 1 & -1\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & 4\\0 & 1 & -1\end{array}\right] \text{ so } c_1 = 4, c_2 = -1.$$

Problem 4: Are there x_1 and x_2 so that $\begin{matrix}x_1 + 3x_2 = 1\\2x_1 + x_2 = 7\end{matrix}$?

Yes. This is the same problem as Problem 3b, just with x_1, x_2 rather than with c_1, c_2.

How are problems 1, 2, 3 and 4 related?

They are all the same problem, just written in different ways! This is really important for you to see and understand. As we have seen, generally we will solve any variation of this problem with the augmented matrix, as we did in Problem 3b or Problem 4.

So we can build $\begin{bmatrix}1\\7\end{bmatrix}$ using $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$. What about building vectors other than $\begin{bmatrix}1\\7\end{bmatrix}$? Rather than consider other vectors one at a time, let's choose a generic vector $\begin{bmatrix}b_1\\b_2\end{bmatrix}$ (where b_1 and b_2 are any number) that represents any vector in R^2.

Problem 5: For <u>any</u> vector $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ in R^2, are there c_1 and c_2 so that $c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$?

$$\left[\begin{array}{cc|c}1 & 3 & b_1\\ 2 & 1 & b_2\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & b_1\\ 0 & -5 & -2b_1+b_2\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & b_1\\ 0 & 1 & \frac{2}{5}b_1-\frac{1}{5}b_2\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & -\frac{1}{5}b_1+\frac{3}{5}b_2\\ 0 & 1 & \frac{2}{5}b_1-\frac{1}{5}b_2\end{array}\right] \Rightarrow \begin{array}{l} c_1 = -\frac{1}{5}b_1+\frac{3}{5}b_2 \\ c_2 = \frac{2}{5}b_1-\frac{1}{5}b_2\end{array}.$$

Of course how to build $\begin{bmatrix}b_1\\b_2\end{bmatrix}$ using $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$ depends on what b_1 and b_2 are. For example, if $b_1 = 1$, $b_2 = 7$, then $\begin{array}{l} c_1 = -\frac{1}{5}\cdot 1+\frac{3}{5}\cdot 7 = 4 \\ c_2 = \frac{2}{5}\cdot 1-\frac{1}{5}\cdot 7 = -1\end{array}$ as found in Problems 1 – 4.

Why would you predict that there is a unique solution to Problem 5 (and all of Problems 1 – 5)?

 # equations = # unknowns, which generally means there will be exactly one solution.

So for any vector $\vec{b}$ in R^2, there are c_1 and c_2 so that $c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\1\end{bmatrix} = \vec{b}$. Any other words, $span\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}\} = R^2$. That is, every vector in R^2 can be written (built) as some linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$.

So for any two vectors v_1 and v_2 in R^2 is it true that $span\{v_1, v_2\} = R^2$? That is, given any two vectors v_1 and v_2 from R^2, can we build any other vector in R^2 using v_1 and v_2?

No. Here is an example that this is not necessarily the case.
(How can we be sure there is no solution? Try to find the solution and see what happens.)

Suppose we want to reach/build points/vectors using $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}-3\\-6\end{bmatrix}$. Given any vector $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ in R^2, can we reach/build the point/vector $\begin{bmatrix}b_1\\b_2\end{bmatrix}$ using $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}-3\\-6\end{bmatrix}$? Let's try.

$\left[\begin{array}{cc|c}1 & -3 & b_1\\ 2 & -6 & b_2\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & b_1\\ 0 & 0 & -2b_1+b_2\end{array}\right]$. We see that there will be a problem (in other words, no solution) if $-2b_1 + b_2 \neq 0$ (Why?). So we can already see that there is a solution only if $-2b_1 + b_2 = 0$, that is, if $b_2 = 2b_1$, that is, $\begin{bmatrix}b_1\\b_2\end{bmatrix} = \begin{bmatrix}b_1\\2b_1\end{bmatrix} = b_1\begin{bmatrix}1\\2\end{bmatrix}$. In other words, only vectors that are some multiple of $\begin{bmatrix}1\\2\end{bmatrix}$ can be reached/built using $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}-3\\-6\end{bmatrix}$. No other vectors can be reached/built.

The **span** (a noun) of a set of vectors is the set of all vectors that can be built using (are linear combinations of) those vectors. If everything in R^3 can be built using $\{\vec{v}_1, \dots, \vec{v}_n\}$, then we say that $\{\vec{v}_1, \dots, \vec{v}_n\}$ **spans** (a verb) R^3. We need at least **3** vectors from R^3 to be able to build all vectors in R^3. See Problem 7.

Problem 7a: Is $\begin{bmatrix}1\\0\\-2\end{bmatrix}$ in $span\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$? **That is, are there c_1 and c_2 such that $c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}1\\0\\-2\end{bmatrix}$?**

Problem 7b: Are there x_1 and x_2 so that $\begin{aligned} x_1 + 4x_2 &= 1 \\ 2x_1 + 5x_2 &= 0 \\ 3x_1 + 6x_2 &= -2 \end{aligned}$?

No. $\left[\begin{array}{cc|c} 1 & 4 & 1 \\ 2 & 5 & 0 \\ 3 & 6 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -5/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & -1 \end{array}\right]$**, and due to the bottom row, there is no solution.**

Why would you predict that there is no solution for Problem 7b (and Problem 7a)?

 # equations > # unknowns, which generally means there will be no solution.

So at least one vector, $\begin{bmatrix}1\\0\\-2\end{bmatrix}$, from R^3 is not in $span\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\}$, so $\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\}$ does not span R^3.

What part/subset of R^3 is $span\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\}$? It looks something like Figure 11 on page 30.

Problem 8a: Are there c_1, c_2 and c_3 so that $c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\4\end{bmatrix} + c_3\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$?

In other words, is $\begin{bmatrix}1\\-2\end{bmatrix}$ in $span\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\right\}$? Yes. See work below.

Problem 8b: Are there x_1, x_2 and x_3 so that $\begin{aligned} x_1 + 3x_2 + 5x_3 &= 1 \\ 2x_1 + 4x_2 + 6x_3 &= -2 \end{aligned}$?

Yes. $\left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 2 & 4 & 6 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -5 \\ 0 & 1 & 2 & 2 \end{array}\right]$ **so** $\begin{aligned} x_1 \quad - x_3 &= -5 \\ x_2 + 2x_3 &= 2 \end{aligned}$ **so** $\begin{aligned} x_1 &= 5 + x_3 \\ x_2 &= 2 - 2x_3. \\ x_3 &= free \end{aligned}$

A few examples: $(x_1, x_2, x_3) = (-5, 2, 0)$ **or** $(-4, 0, 1)$ **or** $(-3, -2, 2)$ **or …**

What do these solutions mean? **Consider the solution $(x_1, x_2, x_3) = (-3, -2, 2)$, which we see satisfies the two equations in Problem 8b. As for being a solution to Problem 8a (which is the same problem as 8b, just in different form), notice that $-3\begin{bmatrix}1\\2\end{bmatrix} - 2\begin{bmatrix}3\\4\end{bmatrix} + 2\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$.**

Why would you guess that there are an infinite number of solutions for Problem 8b (and Problem 8a)?

 # equations < # unknowns, which generally means there will be infinite solutions.

True/**False**. If we have three different vectors from R^3, then they are guaranteed to span R^3.

For example, the vectors $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$ do not span R^3; that is, there are vectors in R^3 that cannot be built using these three vectors. We will show this later. (Notice $\begin{bmatrix}7\\8\\9\end{bmatrix} = 2\begin{bmatrix}4\\5\\6\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix}$.)