Use of technology in row reduction: links on are class homepage.

Most important ideas:

- **The number of solutions of a system of linear equations: 0, 1 or ∞**
- **Related to that: if a system has a solution, is it unique or are there infinite solutions?**
- **Elementary row operations, on page 6.**
- **Before class, make sure that you understand Examples 1, 2 and 3 in the book.**

A linear equation is: **an equation in which the power of each unknown/variable is 1.** Examples:

$$y = \frac{3}{2}x - 5$$

$$3x - 2y = 10$$

$$x_1 - x_2 + 5x_3 + 2x_4 = 7$$

Non-linear equations are equations of any other sort. Examples:

$$y = x^2 + 5$$

$$xy = 30{,}000$$

How are the first two linear equations related? **They are the same equation, that is the same relationship between x and y, just written in different forms.**

$y = \frac{3}{2}x - 5$ is in slope-intercept form: $\frac{3}{2}$ is the slope and -5 is the y −intercept.
$3x - 2y = 10$ is in standard form. The standard form is far more useful to us and the form we will be using almost all of the time this semester.

The third linear equation above shows us what we do if: **we have an equation with several variables—remember they are only so many letters in the alphabet—just keep using x but with subscripts 1, 2, 3, etc.** For example, in some of the (real life) work with digital images, I often use several *hundred thousand* variables.

What is a system of linear equations? **Two or more linear equations.** Examples:

$$\begin{aligned} x + 2y &= 5 \\ 3x + 4y &= 6 \end{aligned} \quad \text{or} \quad \begin{aligned} n + d &= 6 \\ 5n - d &= 0 \\ 5n + 10d &= 75 \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 - 5x_2 + \tfrac{1}{2}x_3 + 2x_4 &= 4 \\ 4x_1 + 3x_2 + \phantom{\tfrac{1}{2}x_3} -7x_4 &= 5 \end{aligned}$$

Example 1a: System of 2 equations with 2 unknowns $\begin{aligned} x + 2y &= 5 \\ 3x + 4y &= 6 \end{aligned}$.

Example 1b: System of 2 equations with 2 unknowns $\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 6 \end{aligned}$.

How are Examples 1a and 1b related? **They are the same problem, just with different names/labels for the two unknowns.**

It really doesn't matter what names\labels we use for the unknowns. What matters—that is, what determines the solution to the problem—is the numbers: the coefficients (the values multiplying the unknowns) and the values on the right hand side.

In Example 1, the coefficient matrix of $\begin{matrix} x + 2y = 5 \\ 3x + 4y = 6 \end{matrix}$ is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and the augmented matrix is $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right]$.

To "augment" means to "add to." It turns out that this simple way of working with systems of equations (by using matrices with numbers in them, rather than the equations themselves with the numbers, letters, equal signs, etc.) is extremely useful and leads to some very powerful theory for solving the kinds of problems in math, science, business, etc. that use linear algebra.

In the solving Example 1, we will see how working with the augmented matrix corresponds to working with the equations. I list three approaches—the first two are really the same thing. The third is the matrix approach and, as you seen, is also equivalent to the first two.

Solve for one variable, substitute into other equation, etc.	Add/subtract one equation to/from another, etc.	Row reduction, also known as Gaussian (or Gauss-Jordan) Elimination	Result
Solve for x in Eqn 1 and substitute into Eqn 2: $\begin{aligned} x + 2y &= 5 \\ -2y &= -9 \end{aligned}$	**Subtract 3*Eqn 1 from Eqn 2:** $\begin{aligned} x + 2y &= 5 \\ -2y &= -9 \end{aligned}$	**Replace Row 1 by 3*Row 1 from Row 2:** $\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 0 & -2 & -9 \end{array}\right]$	**x is eliminated from second equation.**
Divide Eqn 2 by -2: $\begin{aligned} x + 2y &= 5 \\ y &= 9/2 \end{aligned}$	**Divide Eqn 2 by -2:** $\begin{aligned} x + 2y &= 5 \\ y &= 9/2 \end{aligned}$	**Divide Row 2 by -2:** $\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 0 & 1 & 9/2 \end{array}\right]$	**Makes Eqn/Row 2 simpler.**
Substitute value for y from Eqn 2 into Eqn 1: $\begin{aligned} x &= -4 \\ y &= 9/2 \end{aligned}$	**Substitute value for y from Eqn 2 into Eqn 1:** $\begin{aligned} x &= -4 \\ y &= 9/2 \end{aligned}$	**Replace Row 1 with -2*Row 2 from Row 1:** $\left[\begin{array}{cc\|c} 1 & 0 & -4 \\ 0 & 1 & 9/2 \end{array}\right]$	**y is eliminated from second equation.**

How do you know if you have the correct solution? **Substitute the values you found into the original equations. In Example 1, we check that the solutions $x = -4$, $y = 9/2$ satisfy each of the given equations:**

$$-4 + 2(9/2) = 5 \quad \checkmark$$
$$3(-4) + 4(9/2) = 6 \quad \checkmark$$

Terminology: a variable is a value that varies. For example, perhaps your GPA depends on (is a function of) the number of hours per week that you study. It might be that $gpa = 0.5 + 0.1h$ where h is the number of hours per week that you study. In this case, both gpa and h are variables, values that vary. If you study more, your GPA will be higher, and if you study less, your GPA will be lower. In general, if h changes then gpa changes. Sometimes we say that h is the *independent* variable, since it is free to be want it wants: you are free to study as many or few hours as you like. Similarly, gpa is the *dependent* variable, since is varies (and thus is a variable) and its value depends on h. On the other hand, an unknown is simply a value that is fixed but whose value we just don't know yet. It is the value we are trying to determine. So in Example 1, it is better to describe x and y (or x_1 and x_2) as unknowns rather than variables. You might as well get used to using the word "unknown" right away.

Example of Elementary Row Operations
(also known as *Gaussian Elimination* or *Row Reduction*)

The original equations:

$$\begin{array}{rcrcr} 2x & + & 3y & = & 7 \\ -x & + & 4y & = & 2 \end{array}$$

The original matrix:

$$\left[\begin{array}{rr|r} 2 & 3 & 7 \\ -1 & 4 & 2 \end{array}\right]$$

Multiply Eqn 2 by 2:

$$\begin{array}{rcrcr} 2x & + & 3y & = & 7 \\ -2x & + & 8y & = & 4 \end{array}$$

Multiply Row 2 by 2:

$$\left[\begin{array}{rr|r} 2 & 3 & 7 \\ -2 & 8 & 4 \end{array}\right]$$

Add Eqn 1 to Eqn 2:

$$\begin{array}{rcrcr} 2x & + & 3y & = & 7 \\ & & 11y & = & 11 \end{array}$$

Add Row 1 to Row 2:

$$\left[\begin{array}{rr|r} 2 & 3 & 7 \\ 0 & 11 & 11 \end{array}\right]$$

Divide Eqn 2 by 11 (multiply by $\frac{1}{11}$):

$$\begin{array}{rcrcr} 2x & + & 3y & = & 7 \\ & & y & = & 1 \end{array}$$

Divide Row 2 by 11 (multiply by $\frac{1}{11}$):

$$\left[\begin{array}{rr|r} 2 & 3 & 7 \\ 0 & 1 & 1 \end{array}\right]$$

Multiply Eqn 2 by –3:

$$\begin{array}{rcrcr} 2x & + & 3y & = & 7 \\ & & -3y & = & -3 \end{array}$$

Multiply Row 2 by –3:

$$\left[\begin{array}{rr|r} 2 & 3 & 7 \\ 0 & -3 & -3 \end{array}\right]$$

Add Eqn 2 to Eqn 1:

$$\begin{array}{rcrcr} 2x & & & = & 4 \\ & & -3y & = & -3 \end{array}$$

Add Row 2 to Row 1:

$$\left[\begin{array}{rr|r} 2 & 0 & 4 \\ 0 & -3 & -3 \end{array}\right]$$

Divide Eqn 1 by 2 and Eqn 2 by –3:

$$\begin{array}{rcrcr} x & & & = & 2 \\ & & y & = & 1 \end{array}$$

Divide Row 1 by 2 and Row 2 by –3:

$$\left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Example of Gaussian Elimination *with Pivoting*

Initial matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 12 \\ 3 & 5 & 0 & 13 \\ 1 & -2 & 1 & -4 \end{array}\right]$$

$\frac{1}{2}$ **· Row 1**

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 3 & 5 & 0 & 13 \\ 1 & -2 & 1 & -4 \end{array}\right]$$

Row 2 + (–3) · Row 1
Row 3 + (–1) · Row 1

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & -1 & 3 & -5 \\ 0 & -4 & 2 & -10 \end{array}\right]$$

(–1) · Row 2

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & -3 & 5 \\ 0 & -4 & 2 & -10 \end{array}\right]$$

Row 1 + (–2) · Row 2
Row 3 + (+4) · Row 2

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & -4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & -10 & 10 \end{array}\right]$$

$-\frac{1}{10}$ **· Row 3**

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & -4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Row 1 + (–5) · Row 3
Row 2 + 3 · Row 3

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Initial equations

$$\begin{array}{rcrcrcr} 2x & + & 4y & - & 2z & = & 12 \\ 3x & + & 5y & & & = & 13 \\ x & - & 2y & + & z & = & -4 \end{array}$$

$\frac{1}{2}$ **· Eqn 1**

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ 3x & + & 5y & & & = & 13 \\ x & - & 2y & + & z & = & -4 \end{array}$$

Eqn 2 + (–3) · Eqn 1
Eqn 3 + (–1) · Eqn 1

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ & - & y & + & 3z & = & -5 \\ & - & 4y & + & 2z & = & -10 \end{array}$$

(–1) · Eqn 2

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ & & y & - & 3z & = & 5 \\ & - & 4y & + & 2z & = & -10 \end{array}$$

Eqn 1 + (–2) · Eqn 2
Eqn 3 + (+4) · Eqn 2

$$\begin{array}{rcrcrcr} x & & & + & 5z & = & -4 \\ & & y & - & 3z & = & 5 \\ & & & - & 10z & = & 10 \end{array}$$

$-\frac{1}{10}$ **· Eqn 3**

$$\begin{array}{rcrcrcr} x & & & + & 5z & = & -4 \\ & & y & - & 3z & = & 5 \\ & & & & z & = & -1 \end{array}$$

Eqn 1 + (–5) · Eqn 3
Eqn 2 + 3 · Eqn 3

$$\begin{array}{rcrcrcr} x & & & & & = & 1 \\ & & y & & & = & 2 \\ & & & & z & = & -1 \end{array}$$