Math 260 Spring 2025 **Exam 3** **April 15, 2025**

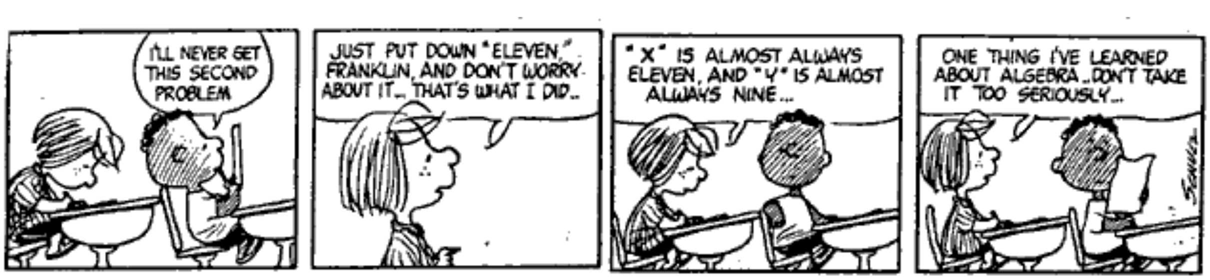
Name: Solutions

Problem	T/F	1	2 / 3 / 4 / 5	Total
Possible	50	24	26	100
Received				

DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 x 5 card of notes, both sides.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.



/50 T/F. Answer the following 25 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may be true some times or under some circumstances, but not always or not necessarily. No explanation is necessary whether true or false.

T **F** If U is square and an orthogonal matrix (i.e., if $U^T U = I$), then it is also true that $UU^T = I$.

U^T is U^{-1}

T **F** If $\vec{x}$ is orthogonal to $\vec{u}$ and $\vec{v}$, then $\vec{x}$ is orthogonal to $\vec{u} - \vec{v}$.

T **F** Given a subspace W of R^n, every vector $\vec{y} \in R^n$ can be written as $\vec{y} = \vec{z}_1 + \vec{z}_2$ where $\vec{z}_1 \in W$ and $\vec{z}_1 \cdot \vec{z}_2 = 0$.

T **F** If $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$, then $\vec{u}$ and $\vec{v}$ are orthogonal.

T **F** For $\vec{u},\ \vec{v} \in R^3$, where $S = span\{\vec{u}, \vec{v}\}$ and $\dim S = 2$, then S is a plane and $S^\perp$ is the line perpendicular to S that passes through the origin.

T **F** Every orthogonal set of vectors is linearly independent.

non-zero

T **F** Every linearly independent set of vectors is orthogonal.

T **F** Given $\vec{y}$ and subspace W, the vector $proj_W\vec{y} - \vec{y}$ is orthogonal to W.

T **F** Where $A = [\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n]$ and $\vec{x} = (x_1, x_2, \dots, x_n)$, in general, the best solution $\vec{x}$ to $A\vec{x} = \vec{b}$ is equivalent to the values of $x_1, x_2, \dots, x_n$ for which the linear combination $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$ is closest to $\vec{b}$.

T **F** If the columns of $\underline{n \times n}$ matrix A are orthonormal, then the solution $\vec{x}$ to $A\vec{x} = \vec{b}$ is $A^{-1}\vec{b}$.

so linearly independent
so A^{-1} exists

T **F** The least squares to solution to $A\vec{x} = \vec{b}$ is the vector $\vec{x}$ which minimizes $\|\vec{b} - A\vec{x}\|$.

T **F** For two vectors $\vec{u}$ and $\vec{v}$ in R^4, $\vec{u} \cdot \vec{v} = 0$ means the vectors are orthogonal.

T **F** $\|\vec{u} - \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$. Not at all.

T F If $\vec{u}$ is in $Col\ A^T$ and if $\vec{v}$ is in $Nul\ A$, then it must be that $\vec{u}\cdot\vec{v} = 0$.

See Theorem 3, page 335.

T F If $W = span\{\vec{v}_1\vec{v}_2\}$, then $proj_W\vec{u} = \frac{\langle\vec{u},\vec{v}_1\rangle}{\langle\vec{v}_1,\vec{v}_1\rangle}\vec{v}_1 + \frac{\langle\vec{u},\vec{v}_2\rangle}{\langle\vec{v}_2,\vec{v}_2\rangle}\vec{v}_2$.

True if are orthogonal.

T F It is possible for four non-zero vectors in R^5 to be mutually orthogonal.

T F For any matrix A, the product $A^T A$ has an inverse. If cols. of A are lin. ind. (which is typical)

T F $Proj_{\vec{a}}\vec{b} = \frac{\vec{b}\cdot\vec{a}}{\vec{b}\cdot\vec{b}}\vec{a}$.

$\vec{a}\cdot\vec{a}$

T F If $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$, then $\|-5f\| = 5\|f\|$.

T F $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ is an orthogonal basis for R^2.

T F If A has orthonormal columns, then for any vectors $\vec{x}$ and $\vec{y}$ it is true that $(A\vec{x})\cdot(A\vec{y}) = \vec{x}\cdot\vec{y}$. See Theorem 7, page 343.

T F Every matrix has a complete (linearly independent) set of orthonormal eigenvectors.

This would be nice. Example: $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ has one e-vector $\begin{bmatrix}1\\0\end{bmatrix}$

T F If A is invertible and has an eigenvalue of 1, then so does A^{-1}.

T F Each eigenvector of A is also an eigenvector of A^2. $A\vec{v} = \lambda\vec{v} \Rightarrow$

$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$.

T F If $A = \begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}7\\8\end{bmatrix}$, then $Proj_{Col\ A}\vec{b} = \begin{bmatrix}7\\8\end{bmatrix}$. since cols. of A span $\mathbb{R}^2$.

T F If $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ is also an eigenvector of both AB and BA.

$AB\vec{v} = A(\lambda_B\vec{v}) = \lambda_B(A\vec{v}) = \lambda_B\lambda_A\vec{v}$

Same for BA.

$$\begin{array}{cc} \vec u_1 & \vec u_2 \end{array}$$

/24 1. Suppose matrix $U = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 5 \\ 1 & 1 \\ 1 & 5 \end{bmatrix}$.

Note that $Col\, U = Col\, A$ (since column 1 of A is twice column 1 of U, and column 2 of A is six times the column 1 of U minus four times column 2 of U).

/7 (a) Find the orthogonal projection of $\vec{b}$ onto $Col\, U$. Notice that the columns of U are orthonormal.

$$\text{Proj}_{Col\,U}\vec b = \frac{\vec b\cdot\vec u_1}{\underbrace{\vec u_1\cdot\vec u_1}_{=1}}\vec u_1 + \frac{\vec b\cdot\vec u_2}{\underbrace{\vec u_2\cdot\vec u_2}_{=1}}\vec u_2 = \frac{3}{2}\begin{bmatrix} \frac12 \\ \frac12 \\ \frac12 \\ \frac12 \end{bmatrix} + \frac12\begin{bmatrix} \frac12 \\ -\frac12 \\ \frac12 \\ -\frac12 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac12 \\ 1 \\ \frac12 \end{bmatrix}$$

/3 (b) Find $UU^T\vec{b}$. $= \text{Proj}_{Col\,U}\vec b =$

UU^T not $= I$ since U is not square

/5 (c) Find the least squares solution $\hat{\vec{x}}$ to $U\vec{x} = \vec{b}$, and then find $U\hat{\vec{x}}$.

$$U^TU = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U^T\vec b = \begin{bmatrix} \frac32 \\ \frac12 \end{bmatrix}$$

$$\text{So } \hat{\vec x} = (U^TU)^{-1}U^T\vec b = \begin{bmatrix} \frac32 \\ \frac12 \end{bmatrix}$$

$$\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & -\frac12 \\ \frac12 & \frac12 \\ \frac12 & -\frac12 \end{bmatrix}\begin{bmatrix} \frac32 \\ \frac12 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac12 \\ 1 \\ \frac12 \end{bmatrix}$$

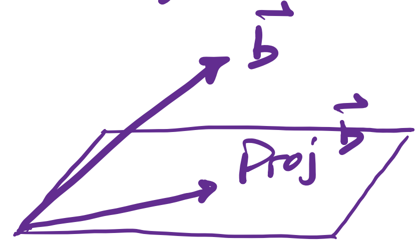
/9 (d) Find the least squares solution $\hat{\vec{x}}$ to $A\vec{x} = \vec{b}$, and then find $A\hat{\vec{x}}$. Same as above.

$$(A^TA)^{-1} = \begin{bmatrix} 4 & 12 \\ 12 & 52 \end{bmatrix}^{-1} = \frac{1}{4\cdot 52 - 12\cdot 12}\begin{bmatrix} 52 & -12 \\ -12 & 4 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix} \qquad A^T\vec b = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\hat{\vec x} = (A^TA)^{-1}A^T\vec b = \frac{1}{16}\begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 18 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac98 \\ -\frac18 \end{bmatrix}$$

$$A\hat{\vec x} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \\ 1 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} \frac98 \\ -\frac18 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac12 \\ 1 \\ \frac12 \end{bmatrix}$$

Same as above since $Col\,A = Col\,U$.



$Col\,A = Col\,U$

/6 2. Where $\vec{x}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{x}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, use the Gram-Schmidt Process to find vectors $\vec{v}_1, \vec{v}_2$ so that $Span\{\vec{v}_1, \vec{v}_2\} = Span\{\vec{x}_1, \vec{x}_2\}$ but where $\vec{v}_1 \cdot \vec{v}_2 = 0$.

$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{6}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$. Easy to see that

/8 3. Given functions $f(t) = 1$ and $g(t) = t$, use the Gram-Schmidt Process to find two functions that are orthogonal under the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$.

$$g - \frac{\langle g, f\rangle}{\langle f, f\rangle} f = t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle} 1 = t - \frac{\int_0^1 t \cdot 1\, dt}{\int_0^1 1 \cdot 1\, dt} \cdot 1 = t - \frac{1}{2}$$

Could check that $\langle 1, t - \frac{1}{2}\rangle = \int_0^1 1(t - \frac{1}{2})\, dt = 0$.

For the final two problems, $A = PDP^{-1} = \begin{bmatrix}1 & -3\\1 & 1\end{bmatrix}\begin{bmatrix}5 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & -3\\1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}3.5 & 1.5\\0.5 & 4.5\end{bmatrix}$.

/6 4. The position $\vec{x}$ of a particle in a planar force field satisfies the equation $\frac{d\vec{x}}{dt} = A\vec{x}$. Where $\vec{x}(0) = \begin{bmatrix}7\\11\end{bmatrix}$, find the function/formula $\vec{x}(t)$ for the particle position at time t.

$\vec{x}(t) = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{5t} + c_2\begin{bmatrix}-3\\1\end{bmatrix}e^{3t}$

$\begin{bmatrix}7\\11\end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}1 & -3\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$ since $e^{5\cdot 0} = 1$, $e^{3\cdot 0} = 1$.

$\Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1 & -3\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}7\\11\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & 3\\-1 & 1\end{bmatrix}\begin{bmatrix}7\\11\end{bmatrix} = \begin{bmatrix}10\\1\end{bmatrix}$.

/6 5. The size of two competing populations $\vec{x}$ change according to $\vec{x}_{k+1} = A\vec{x}_k$. Where $\vec{x}_0 = \begin{bmatrix}7\\11\end{bmatrix}$, find the function/formulation for $\vec{x}_k$.

$\vec{x}_k = c_1\begin{bmatrix}1\\1\end{bmatrix}5^k + c_2\begin{bmatrix}-3\\1\end{bmatrix}3^k$

with same c_1, c_2 as in 4.

Invertible Matrix Theorem for $n \times n$ matrix A

a. A is invertible.
b. A is row equivalent to I.
c. A has n pivot positions.
d. A$\mathbf{x} = \mathbf{0}$ has only trivial solution.
e. Columns of A lin. independent.
f. Linear transf. $\mathbf{x} \to A\mathbf{x}$ 1-to-1.
g. A$\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$.
h. Columns of A span $\mathbf{R}^n$.
i. Linear transf. $\mathbf{x} \to A\mathbf{x}$ onto.
j. There is C such that CA = I.
k. There is D such that AD = I.
l. A^T is invertible.
m. Columns of A form basis for $\mathbf{R}^n$.
n. Column space of A is $\mathbf{R}^n$.
o. dim Col A = n, *i.e.* dimension of column space of A is n.
p. rank A = n, *i.e.* rank of A is n.
q. Nul A = $\{\mathbf{0}\}$, *i.e.* nullspace of A is $\{\mathbf{0}\}$.
r. dim Nul A = 0, the dimension of the null space of A is 0.
s. A has n nonzero eigenvalues, *i.e.* 0 is not an eigenvalue of A.
t. det A $\neq$ 0.
u. $(\text{Col } A)^\perp = \{\mathbf{0}\}$, *i.e.* orthogonal complement of column space of A is $\{\mathbf{0}\}$.
v. $(\text{Nul } A)^\perp = \mathbf{R}^n$, *i.e.* orthogonal complement of null space of A is $\mathbf{R}^n$.
w. Row A = $\mathbf{R}^n$, row space of A is $\mathbf{R}^n$.