Math 260 Spring 2025 **Exam 2** **March 18, 2025**

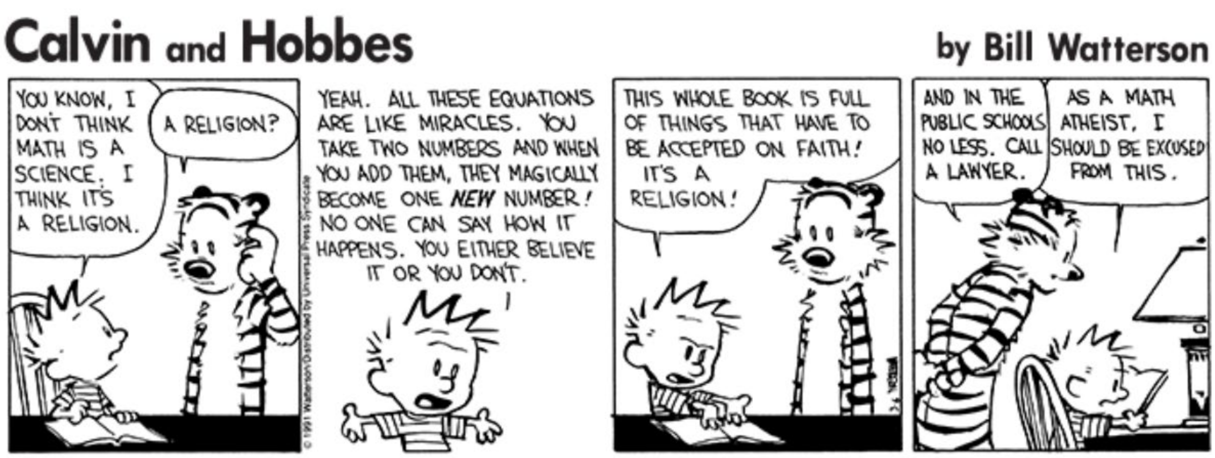
Name: Solutions

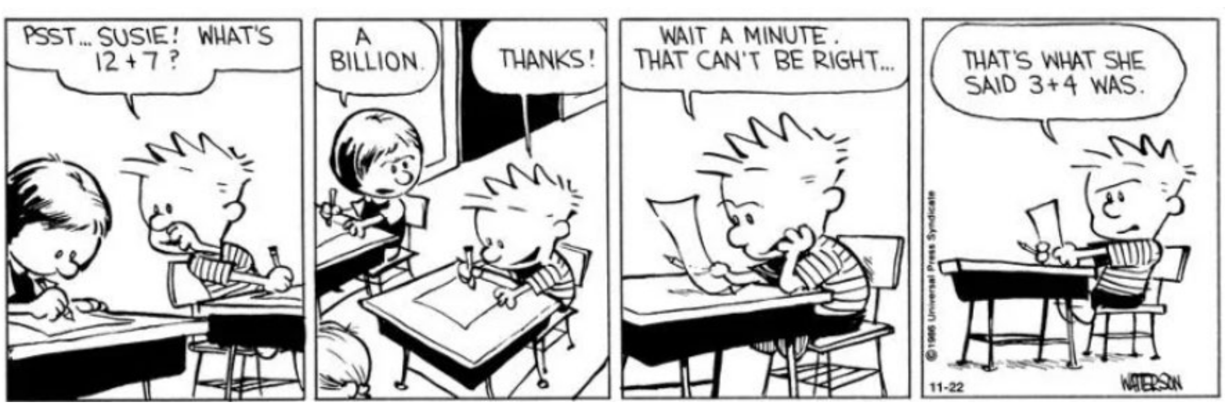
Problem	T/F	1	2 / 3	4 / 5	Total
Possible	46	25	15	14	100
Received					

Do not open your exam until told to do so.

You may use a 3 × 5 card of notes (no calculator).

For full credit, show all work related to finding each solution.





46 points Answer the following True/False questions. Each question is worth 2 points. **"True" means *always true* or *necessarily true*. "False" means that it may be true sometimes or under some circumstances, but not *always* or not *necessarily*.** No explanation is necessary whether true or false.

True False Assuming A^{-1} exists, $\det(A^{-1}B) = \frac{\det(B)}{\det(A)}$.

True False The nullspace of A is the set of all vectors $\vec{x}$ for which $A\vec{0} = \vec{x}$.

True False The columns of any invertible 3×3 matrix form a basis for R^3.

True False If S is a linearly independent set in V, then S is a basis for V.

Example: $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$. V is $\mathbb{R}^3$.

True False If A is a 4×7 matrix (so 4 rows and 7 columns), then $\dim Row\, A < \dim Col\, A$.

Always =

True False If A is a 4×7 matrix, then $4 \leq rank(A) \leq 7$.

$rank(A) \leq 4$ since 4×7

$\Rightarrow$ at most 4 pivot rows.

True False If A is a 4×7 matrix, then $\dim Nul\, A \leq 3$.

≥ 3

There will be at least 3 non-pivot columns.

True False If $A = PDP^{-1}$, where the columns of P are the eigenvectors of A and the entries of diagonal matrix D are the corresponding eigenvalues, then the eigenvectors of A^T are simply the columns of $(P^T)^{-1}$.

True False If $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector of a matrix, then $\begin{bmatrix} -15 \\ -5 \end{bmatrix}$ is also an eigenvector.

True False The largest possible dimension of $Nul\ A$ for a 3×7 matrix A is 3.

At least 4 non-pivot columns so dim Nul A ≥ 4

True False If $A\vec{v} = \lambda\vec{v}$ and if A has an inverse, then $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$.

True False The set of vectors of the form $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$ (for some a and b) is a subspace of R^3.

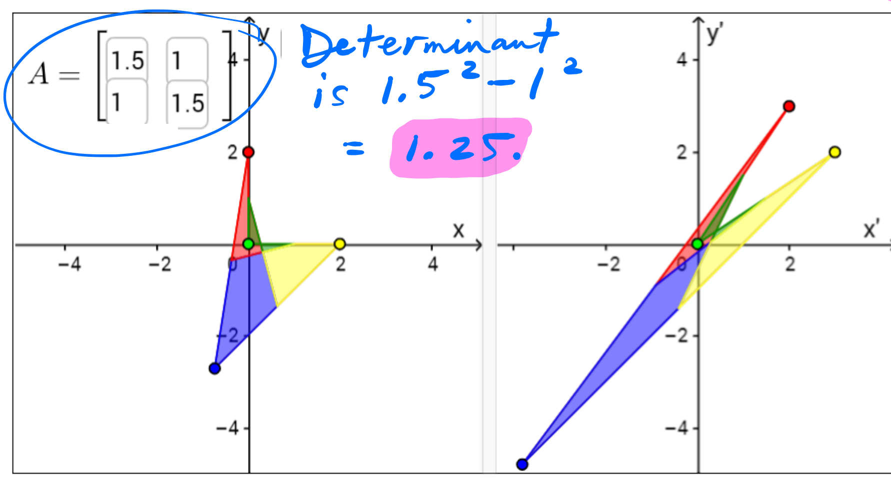
True False Where H is a vector space and $T(H)$ is the image of linear transformation T, then $\dim T(H) \leq \dim H$.

True False If $A\vec{v}_1 = \lambda_1\vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$ where $\vec{v}_1$ and $\vec{v}_2$ are linearly independent eigenvectors of some matrix A, then it must be that $\lambda_1 \neq \lambda_2$.

Can occur for same e-value. A related idea: e-vectors corresponding to different e-vectors will be lin. ind.

True False If the area in the figure at left below is 4, and if the area at right is the area at left transformed by the matrix $\begin{bmatrix} 1.5 & 1 \\ 1 & 1.5 \end{bmatrix}$, then the area at right is 8.

so this would be 1.25(4).



Determinant is $1.5^2 - 1^2$ $= 1.25.$

True False If a 4×4 matrix A has rank 2, then $A\vec{x} = \vec{b}$ has an infinite number of solutions for some right hand side $\vec{b}$, no solution for some other right hand side $\vec{b}$, and a unique solution for yet another right hand side $\vec{b}$.

So A has 2 pivot columns, so 2 non-pivot columns, so if $A\vec{x} = \vec{b}$ has a sol'n, it will have infinite sol'ns (2 free variables) so can't ever have a unique solution.

True False For $\vec{v}_1, \vec{v}_2 \in R^4$, $Span\{v_1, v_2\}$ is a subspace of R^4.

True False For $\vec{v}_1, \vec{v}_2 \in R^4$, $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $Span\{\vec{v}_1, \vec{v}_2\}$.

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}$

True False The sum of three eigenvectors of a matrix is an eigenvector of that same matrix.

Only if they correspond to the same e-value.

True False There is a value of k for which $\begin{bmatrix} 1 & 2 \\ k & 4 \end{bmatrix}$ has an eigenvalue of 0.

Determinant is 0 for $1 \cdot 4 - 2 \cdot k = 0$

$\Leftrightarrow$ "bad" matrix $\Leftrightarrow$ Det. is 0.

True False $\det(-A) = -\det A$.

True False If the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 3$, then $\begin{vmatrix} a + 2c & b + 2d \\ c & d \end{vmatrix} = 6$.

True False Let A be a 5×5 matrix with just three different eigenvalues. It is possible for A to have a complete set of 5 linearly independent eigenvectors.

25 points 1. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$.

/12 Find (and show your work) the eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of A.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 2\cdot 1 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

$= 0 \Rightarrow \lambda = 2, -1.$

E-vectors: $A\vec{x} = \lambda\vec{x} \iff (A - \lambda I)\vec{x} = \vec{0}.$

$$\lambda = 2: \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 - x_2 = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\lambda = -1: \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 + \tfrac{1}{2}x_2 = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

i.e. any multiple of $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$, i.e. $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

/2 Give two different diagonalizations $PDP^{-1} = [\vec{v}_1 \ \vec{v}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [\vec{v}_1 \ \vec{v}_2]^{-1}$ of A.

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} \text{ and } \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

/4 Let B be the R^2 basis with eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Where $\vec{x} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, find $[\vec{x}]_B$, the co-ordinates of $\vec{x}$ with respect to B. (That is, find c_1 and c_2 so that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$.)

$$\begin{bmatrix} 5 \\ -1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}. \quad \text{So } \begin{bmatrix} 5 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

/7 Where $\vec{x} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, find $A^5\vec{x}$. (Note that $2^5 \cdot 3 = 96$.)

$$A^5\vec{x} = A^5\left(3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = 2^5 \cdot 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1)^5(-2)\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 96 \\ 96 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 94 \\ 100 \end{bmatrix}. \quad \text{Or find } A^5\vec{x}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2^5 & 0 \\ 0 & (-1)^5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} = \cdots = \begin{bmatrix} 21 & 11 \\ 22 & 10 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 94 \\ 100 \end{bmatrix}.$$

5 points 2. <u>Use Cramer's Rule</u> to find x_3 (not x_1 or x_2) in the linear system
$$\begin{aligned} 3x_1 \qquad\quad &= 6 \\ 4x_1 + 2x_2 \quad &= 2 \\ 5x_1 \qquad + x_3 &= 11 \end{aligned}.$$

$$x_3 = \frac{\begin{vmatrix} 3 & 0 & 6 \\ 4 & 2 & 2 \\ 5 & 0 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 0 & 1 \end{vmatrix}} = \frac{2\begin{vmatrix} 3 & 6 \\ 5 & 11 \end{vmatrix}}{3 \cdot 2 \cdot 1} = \frac{2(3 \cdot 11 - 5 \cdot 6)}{6} = 1.$$

10 points 3. Consider probability matrix $P = \begin{bmatrix} .2 & .4 \\ .8 & .6 \end{bmatrix}$ and initial vector $\vec{x}_0 = \begin{bmatrix} 0 \\ 70 \end{bmatrix}$. Let $\vec{x}_k = P^k \vec{x}_0$.

/2 Find $\vec{x}_1$. $= \begin{bmatrix} .2 & .4 \\ .8 & .6 \end{bmatrix}\begin{bmatrix} 0 \\ 70 \end{bmatrix} = \begin{bmatrix} 28 \\ 42 \end{bmatrix}$

/6 Find $\vec{x}_\infty = \vec{q}$ where $P\vec{q} = \vec{q} \Leftrightarrow (P - I)\vec{q} = \vec{0}$.

$$\left[\begin{array}{cc|c} -.8 & .4 & 0 \\ .8 & -.4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad x_1 - \tfrac{1}{2}x_2 = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

1:2 ratio for $x_1 : x_2$.

So with sum of values of 70, we have $70\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$.

/1 Find the eigenvector of P that corresponds to eigenvalue 1.

/1 Find P^∞.

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

8 points 4. Consider matrix A which is row equivalent to matrix B:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & -2 & 4 & 2 & 5 \\ 1 & 1 & 1 & 3 & 6 \\ 1 & 3 & -1 & 4 & 7 \\ 1 & 2 & 0 & 5 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

/2 Find a basis for $Col\ A$:

Columns 1, 2, 4 of A.

/2 Find a basis for $Row\ A$:

First three rows of A or B.

/4 Find a basis for $Nul\ A$:

$$\begin{aligned} x_1 \quad + 2x_3 \quad + 3x_5 &= 0 \\ x_2 - x_3 \quad &= 0 \\ x_4 + x_5 &= 0 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The basis.

6 points 5. Prove that the nullspace of a $m \times n$ matrix is a subspace of R^n.

If $\vec{x}_1, \vec{x}_2 \in Nul\ A$ then $A\vec{x}_1 = 0$, $A\vec{x}_2 = 0$.

Then $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$

$\Rightarrow \vec{x}_1 + \vec{x}_2 \in Nul\ A$.

Also, $A(c\vec{x}_1) = cA\vec{x}_1 = c\vec{0} = \vec{0}$

$\Rightarrow c\vec{x}_1 \in Nul\ A$.

Also, $A\vec{0} = \vec{0} \Rightarrow \vec{0} \in Nul\ A$.

Invertible Matrix Theorem for $n \times n$ matrix A

a. A is invertible.

b. A is row equivalent to I.

c. A has n pivot positions.

d. A$\mathbf{x}$ = $\mathbf{0}$ has only trivial solution.

e. Columns of A lin. independent.

f. Linear transf. $\mathbf{x} \to A\mathbf{x}$ 1-to-1.

g. A$\mathbf{x}$ = $\mathbf{b}$ has at least one solution for each $\mathbf{b}$.

h. Columns of A span $\mathbf{R}^n$.

i. Linear transf. $\mathbf{x} \to A\mathbf{x}$ onto.

j. There is C such that CA = I.

k. There is D such that AD = I.

l. A^T is invertible.

m. Columns of A form basis for $\mathbf{R}^n$.

n. Column space of A is $\mathbf{R}^n$.

o. dim Col A = n, *i.e.* dimension of column space of A is n.

p. rank A = n, *i.e.* rank of A is n.

q. Nul A = $\{\mathbf{0}\}$, *i.e.* nullspace of A is $\{\mathbf{0}\}$.

r. dim Nul A = 0, the dimension of the null space of A is 0.

s. A has n nonzero eigenvalues, *i.e.* 0 is not an eigenvalue of A.

t. det A $\neq$ 0.

u. $(\text{Col } A)^{\perp} = \{\mathbf{0}\}$, *i.e.* orthogonal complement of column space of A is $\{\mathbf{0}\}$.

v. $(\text{Nul } A)^{\perp} = \mathbf{R}^n$, *i.e.* orthogonal complement of null space of A is $\mathbf{R}^n$.

w. Row A = $\mathbf{R}^n$, row space of A is $\mathbf{R}^n$.