Math 260 Spring 2025	Midterm Exam 1	February 13, 2025

Name: Solutions

Problem	T / F	1 / 2	3 / 4	5 / 6	7 / 8	Total
Possible	34	20	15	16	15	100
Received						

DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 x 5 card (both sides) of notes, but no calculator.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.

"Okee-doke! Let's just double-check. We're 130 feet up and we've got 45 yards of bungee cord, that's uh ... 90 feet. Allow for 30 feet of stretching, that gives us a total of ...120 feet. Perfect!"

34 points T/F. Answer the following 18 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may sometimes be true, but not always or not necessarily. **No work or explanation or justification is needed for these questions—just circle either True or False.**

True False If the columns of a matrix are linearly independent, then it is possible for the problem $A\vec{x} = \vec{b}$ to have more than one solution $\vec{x}$ for some right-hand side $\vec{b}$.

True False If the columns of an $m \times n$ matrix span R^m and are linearly independent, then it must be that $m = n$.

True False If $A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$, then $A\vec{x} = \vec{b}$ will have multiple (infinite actually) solutions for all $\vec{b}$. True if $A\vec{x} = \vec{b}$ has a solution, but not neccesarily for all $\vec{b}$.

True False If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$.

True False If $A = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 0 & 4 \\ 1 & 4 & 1 \end{bmatrix}$, then $A\vec{x} = \vec{0}$ must have more than one solution.

True False The columns of a 2×3 matrix (so 2 rows, 3 columns) could span R^3. $\mathbb{R}^2$

True False The columns of a 2×3 must span R^2. Example: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

True False The columns of a 2×3 matrix must be linearly dependent.
At most 2 pivots, so at least one extra (non-pivot) column.

True False If A is a 4×3 matrix and B is a 3×2 matrix, then $(AB)^T(AB)$ is a 2×2 matrix.

True False The problem of solving for x_1 and x_2 in the system of equations

$$3x_1 + 2x_2 = 1$$
$$6x_1 + 4x_2 = 3$$

is equivalent of solving for x_1 and x_2 in the vector equation

$$x_1 \begin{bmatrix} 3 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

True False If vector $\vec{b} = A\vec{x}$ for some vector $\vec{x}$, then $\vec{b}$ is a linear combination of the columns of A.

True False If the columns of $n \times n$ A do not span R^n, then the columns are linearly dependent.

So A is "bad"

True False If $\underbrace{Span\{\vec{v}_1, \vec{v}_2\} = Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

So $\vec{v}_3, \vec{v}_4$ are linear combinations of $\vec{v}_1, \vec{v}_2$.

True False In R^3, if $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then $Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a plane in R^3.

$= Span\{\vec{v}_1, \vec{v}_2\} = \nearrow$

True False $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ can be written as a linear combination of $\underbrace{\left\{\begin{bmatrix} 2.41 \\ -3.15 \end{bmatrix}, \begin{bmatrix} 5.6 \\ 6.5 \end{bmatrix}\right\}}$.

lin. ind. $\Rightarrow$ span $\mathbb{R}^2$

True False If the columns of $n \times n$ matrix A are linearly dependent, then $A\vec{x} = \vec{b}$ <u>might</u> have a solution, depending on what $\vec{b}$ is.

Example: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

True False For $A\vec{x} = \vec{b}$, it is possible that $A\vec{x} = \vec{b}_1$ has an <u>infinite</u> number of solutions for some $\vec{b}_1$ while $A\vec{x} = \vec{b}_2$ has <u>one</u> solution for some other $\vec{b}_2$.

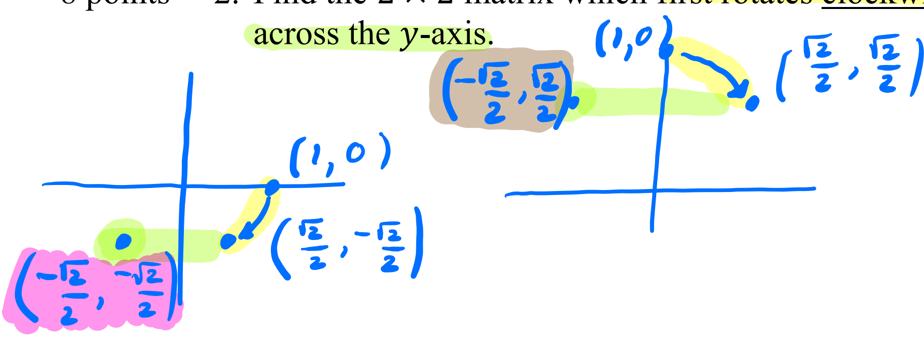
$\Rightarrow$ any system with a solution has infinite solutions.

12 points **1.** Suppose I have some nickels (5 cents each) and dimes (10 cents each). I have 13 coins total, I have 3 more nickels than dimes (so $n = d + 3$), and I have 90 cents total. How many of each type of coin do I have? Solve this by coming up with the three equations that correspond to these three conditions (13 coins total, 3 more nickels than dimes, and 90 cents total), then doing row reduction (i.e. Gaussian Elimination) to find the solution(s) to this system of equations. <u>Don't just guess the solution</u>. Or show that there is no solution, if that is the case.

$$n + d = 13$$
$$n - d = 3$$
$$5n + 10d = 90$$

$$\left[\begin{array}{cc|c} 1 & 1 & 13 \\ 1 & -1 & 3 \\ 5 & 10 & 90 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow n \\ \leftarrow d \\ \end{array}$$

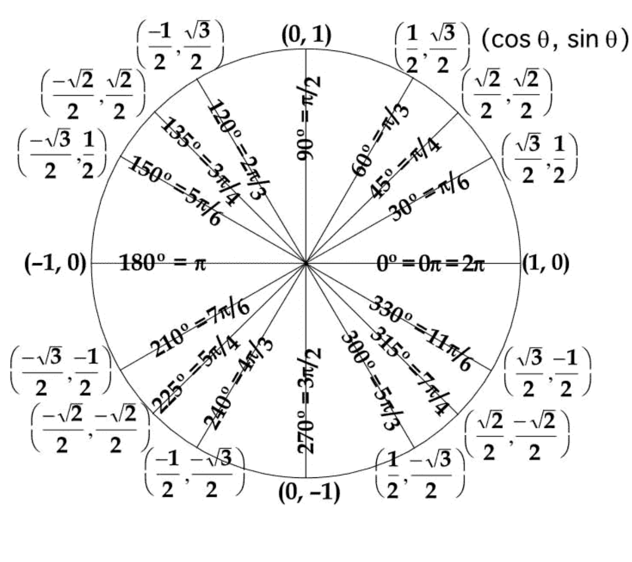
8 points 2. Find the 2×2 matrix which first rotates <u>clockwise</u> ↻ by 45 degrees, and then reflects across the y-axis.



$$\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Or the two steps separately:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} =$$



10 points **3.** Solve for $x,\ y$ and z in the following system by <u>finding the inverse</u> of its coefficient matrix

$$\begin{aligned} x + y + \ \ z &= \ \ 1 \\ 2x + y - \ \ z &= \ \ 5 \\ x + y + 2z &= -1 \end{aligned} \qquad\qquad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

and using it to find the values of x, y and z. Use the method $[\,A \mid I\,] \rightarrow [\,I \mid A^{-1}]$ for finding the inverse. You should not encounter any fractions in finding it. Show work. Don't just guess answers.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \cdots \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 1 & 2 \\ 0 & 1 & 0 & 5 & -1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ \downarrow \ \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$

5 points **4.** Prove that if the columns of A are linearly independent, then the columns of A^2 are linearly independent. (Recall that a matrix M has linearly independent columns if $M\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$.)

$A^2\vec{x} = \vec{0} \Rightarrow A(A\vec{x}) = \vec{0} \Rightarrow A\vec{x} = \vec{0}$ since

$\Rightarrow \vec{x} = \vec{0}$ since

$\Rightarrow$ Cols. of A^2 are lin. dep.

6 points **5.** We are interested in solving the following system of equations,

$$2x + 3y = 7$$
$$8x + ay = b$$

where a and b are some constants whose values have not yet been decided. Give an example of values of a and b that result in the system having:

No solution: $a = 12$ $b = 28$

One solution: $a \neq 12$ $b =$ any number

Infinite solutions: $a = 12$ $b = 28$

10 points **6.** Find the solution(s) to each of the following linear systems. If a system has more than one solution, give the general solution and then give *at least two* specific solutions. If a system has no solution, state that. Notice the left hand side is the same in both.

$$x + y - z + 2w = 5$$
$$-x - y + 3z = 7$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 5 \\ -1 & -1 & 3 & 0 & 7 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 11 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right]$$

$x = -y - 3w + 11$
$y = y$
$z = -w + 6$
$w = w$

so $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 11 \\ 0 \\ 6 \\ 0 \end{bmatrix}$, eg. $\begin{bmatrix} 10 \\ 1 \\ 6 \\ 0 \end{bmatrix}$

$y = 0$, $w = 0$ $\qquad$ $y = 1$, $w = 0$

$$x + y - z + 2w = 0$$
$$-x - y + 3z = 0$$

$y\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, eg. $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$y = 1$, $w = 0$ $\qquad$ $y = 0$, $w = 1$

10 points **7.** A company produces two items, but uses up some of each product in the production process, as described by the input-output (consumption) matrix

$$C = \begin{bmatrix} .5 & 0 \\ .2 & .6 \end{bmatrix}.$$

Note for this problem that $(.5)(.4) = .2$, and that $\frac{.5}{.2} = \frac{5}{2}$ and $\frac{.4}{.2} = 2$.

How much would you need to produce in order to *end up* with 10 units of each product? (Use the **formula** for finding the 2×2 matrix in this problem.) What is one thing about your solution that makes you think it is reasonable, i.e. that it could be the correct answer?

$$(I-C)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .5 & 0 \\ .2 & .6 \end{bmatrix}\right) = \begin{pmatrix} .5 & 0 \\ -.2 & .4 \end{pmatrix}^{-1}$$

$$= \frac{1}{\underbrace{(.5)(.4)-(-.2)(0)}_{.2}} \begin{pmatrix} .4 & 0 \\ .2 & .5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & \frac{5}{2} \end{pmatrix}$$

$$\vec{x} = \begin{bmatrix} 2 & 0 \\ 1 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 35 \end{bmatrix} > \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

5 points **8.** For what value(s) of a are the vectors $\begin{bmatrix} a \\ 1 \end{bmatrix}, \begin{bmatrix} a^2 \\ a^3 \end{bmatrix}$ linearly **dependent**?

$$\begin{bmatrix} a & a^2 \\ 1 & a^3 \end{bmatrix} \xrightarrow{\frac{1}{a} R1} \begin{bmatrix} 1 & a \\ 1 & a^3 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & a \\ 0 & a^3-a \end{bmatrix}$$

If $a \neq 0$.

If $a = 0$, vector two is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Lin. dep. if $a^3 - a = 0$

$a(a^2-1) = 0$

$a = \pm 1$

Or use determinants: $\begin{vmatrix} a & a^2 \\ 1 & a^3 \end{vmatrix} = a^4 - a^2 = a^2(a^2-1) = 0$

$\Rightarrow a = 0$ or ± 1.

Invertible Matrix Theorem for $n \times n$ matrix A

a. A is invertible.
b. A is row equivalent to I.
c. A has n pivot positions.
d. $A\mathbf{x} = \mathbf{0}$ has only trivial solution.
e. Columns of A lin. independent.
f. Linear transf. $\mathbf{x} \to A\mathbf{x}$ 1-to-1.
g. $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$.
h. Columns of A span $\mathbf{R}^n$.
i. Linear transf. $\mathbf{x} \to A\mathbf{x}$ onto.
j. There is C such that $CA = I$.
k. There is D such that $AD = I$.
l. A^T is invertible.
m. Columns of A form basis for $\mathbf{R}^n$.
n. Column space of A is $\mathbf{R}^n$.
o. dim Col A = n, *i.e.* dimension of column space of A is n.
p. rank A = n, *i.e.* rank of A is n.
q. Nul A = $\{\mathbf{0}\}$, *i.e.* nullspace of A is $\{\mathbf{0}\}$.
r. dim Nul A = 0, the dimension of the null space of A is 0.
s. A has n nonzero eigenvalues, *i.e.* 0 is not an eigenvalue of A.
t. det A $\neq$ 0.
u. $(\text{Col } A)^{\perp} = \{\mathbf{0}\}$, *i.e.* orthogonal complement of column space of A is $\{\mathbf{0}\}$.
v. $(\text{Nul } A)^{\perp} = \mathbf{R}^n$, *i.e.* orthogonal complement of null space of A is $\mathbf{R}^n$.
w. Row A = $\mathbf{R}^n$, row space of A is $\mathbf{R}^n$.