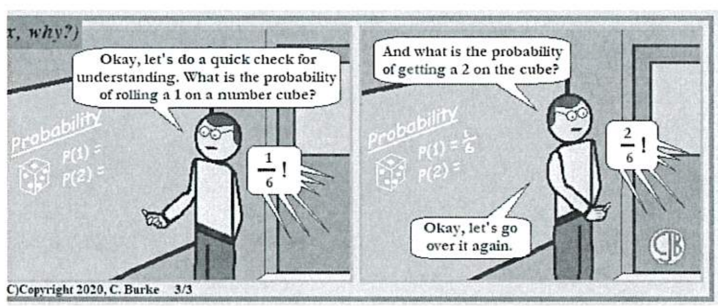
Math 260 Spring 2024 — Exam 3 — April 9, 2024

Name: Solutions

Problem	T/F	1	2	3	4 / 5	Total
Possible	30	10	26	20	14	100
Received						

YOU MAY USE A 3 X 5 CARD OF HANDWRITTEN NOTES, BOTH SIDES, AND A CALCULATOR.

SHOW PERTINENT WORK IN SOLVING EACH PROBLEM.



"If there are 7 cats in a sack and I draw one at random,..."

"... what is the probability that I will draw you?"

30 points T/F. Answer the following 15 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may be true sometimes or under some circumstances, but not always or not necessarily. <u>No explanation is necessary</u> whether true or false.

(1) T F W is orthogonal to $W^{\perp}$ means each vector in W is orthogonal to at least one vector in $W^{\perp}$.

all !

(2) T F If $W = span\{\vec{v}_1 \vec{v}_2\}$, then $Proj_W \vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2$.

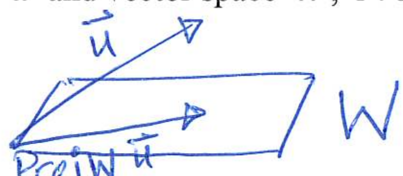
Only if $\vec{v}_1, \vec{v}_2$ are mutually orthogonal.

(3) T F If every column of $\boxed{\text{square}}$ matrix A is orthogonal to every other column of A and each column is of length/size 1, then $A A^T = I$.

A orthogonal $\Rightarrow A^T A = I$

$\Rightarrow A A^T = I$, since A is square.

(4) T F For vector $\vec{u}$ and vector space W, $Proj_W \vec{u}$ is orthogonal to $\vec{u}$.



(5) T F It is possible for six non-zero vectors in R^5 to be mutually orthogonal.

Mutually orthogonal and non-zero $\Rightarrow$ lin. ind.,
but cannot have 6 lin. ind. vectors in $\mathbb{R}^5$.

(6) T F For vector space W, if $Proj_W \vec{u} = \vec{u}$, then $\vec{u}$ is a vector in W.

(7) T F Every orthonormal set of vectors is linearly independent.

$\Rightarrow$ all vectors are length 1

$\Rightarrow$ no $\vec{0}$ vectors

(and since orthogonal)

$\Rightarrow$ lin. independent

(8) T F The best (i.e. least squares) solution to $A\vec{x} = \vec{b}$ is the vector $\vec{x}$ for which $\|\vec{b} - A\vec{x}\|$ is minimized.

(9) T F If the columns of $\boxed{n \times n}$ matrix A are orthonormal (i.e. if $A^T A = I$), then the exact solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^T \vec{b}$.

Square, orthonormal columns $\Rightarrow A^T$ is A^{-1}

(10) T F If $\vec{u}$ is in $Col\, A^T$ and if $\vec{v}$ is in $Nul\, A$, then it must be that $\vec{u} \cdot \vec{v} = 0$.

(11) T F $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for R^3.

These are not mutually orthogonal.

(12) T F For vector space W, if $\vec{x} = \vec{y} + \vec{z}$ where $\vec{y} \in W$ and $\vec{z} \in W^{\perp}$, then $\|\vec{x}\|^2 = \|\vec{y}\|^2 + \|\vec{z}\|^2$.

See Theorem 2, page 334.

(13) T F If the columns of $m \times n$ U are orthonormal, then the rows of U are also orthonormal.

Would be true if U is square, i.e. if $m = n$.

(14) T F Suppose $\vec{u}, \vec{v} \in R^{10}$ form a basis for W, so $\dim W = 2$. Then the set of all vectors that are orthogonal to W is a subspace of R^8.

is an 8-dimensional subspace of $\mathbb{R}^{10}$.

(15) T F Suppose $\hat{\vec{b}} = Proj_W \vec{b}$. Then $\vec{b} = Proj_W \hat{\vec{b}}$.

10 points 1. We'll find a 2×2 matrix by looking at what the matrix and its inverse do to vectors.

Given initial vector $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $\vec{x}_{k+1} = A\vec{x}_k$, we have (approximately)

1

k	0	1	$\cdots$	10	11
$\vec{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 14 \\ 18 \end{bmatrix}$	$\cdots$	$\begin{bmatrix} 1.5 \times 10^9 \\ 2.0 \times 10^9 \end{bmatrix}$	$\begin{bmatrix} 1.5 \times 10^{10} \\ 2.0 \times 10^{10} \end{bmatrix}$

and (now using $\vec{x}_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and A^{-1}) where $\vec{x}_{k+1} = A^{-1}\vec{x}_k$ we have (approximately)

2

k	0	1	$\cdots$	51	52
$\vec{x}_k$	$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}$	$\cdots$	$\begin{bmatrix} 4.44 \times 10^{-16} \\ 8.88 \times 10^{-16} \end{bmatrix}$	$\begin{bmatrix} 2.22 \times 10^{-16} \\ 4.44 \times 10^{-16} \end{bmatrix}$

Find A. Hint: first, estimate the eigenvalues and eigenvectors, and then use these to write A as the produce of three matrices. Note: you are welcome to leave A written as the product of three matrices and NOT actually do the computation to find A.

1 $\Rightarrow$ A has e-value/vector $10, \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$

2 $\Rightarrow$ A^{-1} has e-value/vector $\frac{1}{2}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow$ A has e-value/vector $2, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

So $A = \begin{bmatrix} 1.5 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1.5 & 1 \\ 2 & 2 \end{bmatrix}^{-1}$

26 points 2. Suppose $A = \begin{bmatrix}1 & 2\\ 1 & 3\\ 1 & 4\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}4\\ -3\\ 2\end{bmatrix}$.

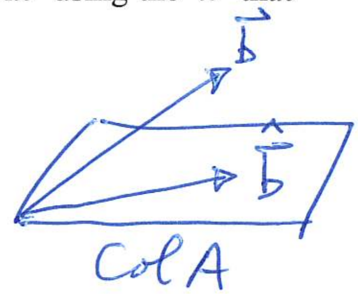
/8 (a) Find the least squares solution $\hat{\vec{x}}$ to $A\vec{x} = \vec{b}$. Show appropriate work.

$$A^TA = \begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 3\\ 1 & 4\end{bmatrix} = \begin{bmatrix}3 & 9\\ 9 & 29\end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix}3\\ 7\end{bmatrix}$$

$$\text{So } \hat{\vec{x}} = \underbrace{\frac{1}{3\cdot 29 - 9\cdot 9}\begin{bmatrix}29 & -9\\ -9 & 3\end{bmatrix}}_{(A^TA)^{-1}}\begin{bmatrix}3\\ 7\end{bmatrix} = \frac{1}{6}\begin{bmatrix}87-63\\ -27+21\end{bmatrix} = \begin{bmatrix}4\\ -1\end{bmatrix}$$

/2 (b) Find $\hat{\vec{b}}$, the projection of $\vec{b}$ onto $Col\,A$. That is, compute $\hat{\vec{b}} = A\hat{\vec{x}}$ using the $\hat{\vec{x}}$ that you found in (a).

$$\hat{\vec{b}} = A\hat{\vec{x}} = \begin{bmatrix}1 & 2\\ 1 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}4\\ -1\end{bmatrix} = \begin{bmatrix}2\\ 1\\ 0\end{bmatrix}$$



Notice: $\vec{b} - \hat{\vec{b}} = \begin{bmatrix}2\\ -4\\ 2\end{bmatrix}$ is $\perp$ to columns of A

/5 (c) Next, use the Gram-Schmidt Process to find an orthogonal basis (call these vectors $\vec{v}_1, \vec{v}_2$) for $Col\,A$. That is, find $\vec{v}_1, \vec{v}_2$ from the two columns of A. Show appropriate work.

$$\vec{v}_1 = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix}2\\ 3\\ 4\end{bmatrix} - \frac{\begin{bmatrix}2\\ 3\\ 4\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}}{\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$= \begin{bmatrix}2\\ 3\\ 4\end{bmatrix} - \frac{9}{3}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

Problem 2 is continued on this page

/5 (d) Find the projection $Proj_{Span\{\vec{v}_1,\vec{v}_2\}}\vec{b}$ for the orthogonal $\vec{v}_1, \vec{v}_2$ that you found in (c).

$$\hat{\vec{b}} = \frac{\vec{b}\cdot\vec{V}_1}{\vec{V}_1\cdot\vec{V}_1}\vec{V}_1 + \frac{\vec{b}\cdot\vec{V}_2}{\vec{V}_2\cdot\vec{V}_2}\vec{V}_2$$

$$= \frac{3}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{-2}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix}$$

(same as in (b))

/6 (e) Finally, using the work that you did in (c), find the QR factorization of A, where Q has ortho<u>normal</u> columns and R is upper triangular.

Normalize $\begin{bmatrix}1 & -1\\1 & 0\\1 & 1\end{bmatrix}$ to get $Q = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{3}} & 0\\\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\end{bmatrix}$

So $Q^TQ = I$ and $QR = A$

$$\Rightarrow R = Q^TA = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\\frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 3\\1 & 4\end{bmatrix}$$

$$= \begin{bmatrix}\frac{3}{\sqrt{3}} & \frac{9}{\sqrt{3}}\\0 & \frac{2}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\sqrt{3} & 3\sqrt{3}\\0 & \sqrt{2}\end{bmatrix}.$$

20 points 3. Suppose $\vec{x} = \begin{bmatrix}1\\-4\\-3\\5\end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}$, and $W = span\{\vec{u}_1, \vec{u}_2\}$.

/7 (a) Find vectors $\vec{y} \in W$ and $\vec{z} \in W^T$ so that $\vec{x} = \vec{y} + \vec{z}$. (Notice that $\vec{u}_1 \perp \vec{u}_2$.)

$$\vec{y} = \text{Proj}_W \vec{x} = \frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 = \frac{9}{3}\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} + \frac{6}{6}\begin{bmatrix}4\\-2\\-2\\3\end{bmatrix} = \begin{bmatrix}4\\-2\\-2\\3\end{bmatrix}$$

Then $\vec{z} = \vec{x} - \vec{y} = \begin{bmatrix}1\\-4\\-3\\5\end{bmatrix} - \begin{bmatrix}4\\-2\\-2\\3\end{bmatrix} = \begin{bmatrix}-3\\-2\\-1\\2\end{bmatrix}$, which $\perp$ u_1, u_2

/1 (b) Briefly explain why $\dim W^{\perp} = 2$.

/6 (c) Find two vectors that form a basis for $W^{\perp}$. (You might want to first double check that your two vectors are orthogonal to $\vec{u}_1, \vec{u}_2$.)

Need $\vec{x}\cdot\vec{u}_1 = 0$
$\vec{x}\cdot\vec{u}_2 = 0$

$$\begin{bmatrix}1 & 0 & -1 & 1\\1 & -2 & 1 & 0\end{bmatrix}\overbrace{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}^{\vec{x}} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1 & 0 & -1 & 1 & 0\\1 & -2 & 1 & 0 & 0\end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c}1 & 0 & -1 & 1 & 0\\0 & 1 & -1 & \frac{1}{2} & 0\end{array}\right]$$

$x_1 = x_3 - x_4$
$x_2 = x_3 - \frac{1}{2}x_4$
$x_3 = x_3$
$x_4 = x_4$

So $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\-\frac{1}{2}\\0\\1\end{bmatrix}$

/6 (d) Show that the vector $\vec{z}$ that you found in (a) is indeed in $W^{\perp}$. Do this by finding how you would build (as a linear combination) $\vec{z}$ from the two vectors you found in (c).

Need: $\begin{bmatrix}-3\\-2\\-1\\2\end{bmatrix} = c_1\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}-1\\-\frac{1}{2}\\0\\1\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & -1 & -3\\1 & -\frac{1}{2} & -2\\1 & 0 & -1\\0 & 1 & 2\end{array}\right] \sim \left[\begin{array}{cc|c}1 & -1 & -3\\0 & \frac{1}{2} & 1\\0 & 1 & 2\\0 & 1 & 2\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\\0 & 0 & 0\end{array}\right]\begin{array}{l}\leftarrow c_1\\\leftarrow c_2\\ \\ \end{array}$$

For the final two problems, $A = PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -3 \\ -6 & 0 \end{bmatrix}$.

8 points 4. The sizes of two competing populations $\vec{x}$ change according to $\vec{x}_{k+1} = A\vec{x}_k$.

Where $\vec{x}_0 = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$, find $\vec{x}_1$. In general, what is $\vec{x}_k$?

$$\vec{x}_k = c_1 (-3)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 (6)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{where}$$

$$\begin{bmatrix} 1 \\ 11 \end{bmatrix} = \vec{x}_0 = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

points 5. The position $\vec{x}$ of a particle in a planar force field satisfies the equation $\frac{d\vec{x}}{dt} = A\vec{x}$.

Where $\vec{x}(0) = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$, find the position of the particle at time t.

$$\vec{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

same c_1, c_2 as found above.