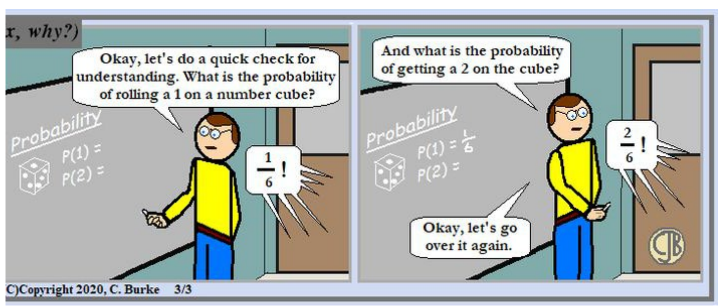
Math 260 Spring 2024 **Exam 3** **April 9, 2024**

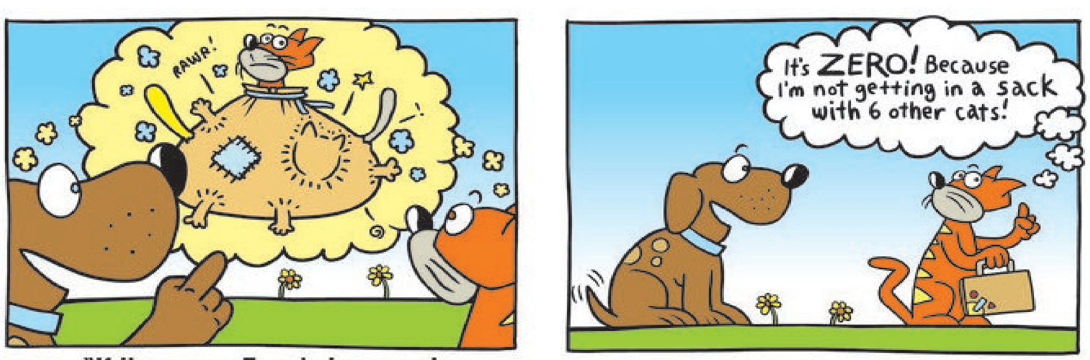
Name: ______________________________________

Problem	T/F	1	2	3	4 / 5	Total
Possible	30	10	26	20	14	100
Received						

YOU MAY USE A 3 X 5 CARD OF HANDWRITTEN NOTES, BOTH SIDES, AND A CALCULATOR.

SHOW PERTINENT WORK IN SOLVING EACH PROBLEM.





"If there are 7 cats in a sack and I draw one at random,..."

"...what is the probability that I will draw you?"

30 points T/F. Answer the following 15 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may be true sometimes or under some circumstances, but not always or not necessarily. No explanation is necessary whether true or false.

(1) T F W is orthogonal to $W^{\perp}$ means each vector in W is orthogonal to at least one vector in $W^{\perp}$.

(2) T F If $W = span\{\vec{v}_1 \vec{v}_2\}$, then $Proj_W \vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2$.

(3) T F If every column of square matrix A is orthogonal to every other column of A and each column is of length/size 1, then $A A^T = I$.

(4) T F For vector $\vec{u}$ and vector space W, $Proj_W \vec{u}$ is orthogonal to $\vec{u}$.

(5) T F It is possible for six non-zero vectors in R^5 to be mutually orthogonal.

(6) T F For vector space W, if $Proj_W \vec{u} = \vec{u}$, then $\vec{u}$ is a vector in W.

(7) T F Every orthonormal set of vectors is linearly independent.

(8) T F The best (i.e. least squares) solution to $A\vec{x} = \vec{b}$ is the vector $\vec{x}$ for which $||\vec{b} - A\vec{x}||$ is minimized.

(9) T F If the columns of $n \times n$ matrix A are orthonormal (i.e. if $A^T A = I$), then the exact solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^T \vec{b}$.

(10) T F If $\vec{u}$ is in $Col\,A^T$ and if $\vec{v}$ is in $Nul\,A$, then it must be that $\vec{u} \cdot \vec{v} = 0$.

(11) T F $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for R^3.

(12) T F For vector space W, if $\vec{x} = \vec{y} + \vec{z}$ where $\vec{y} \in W$ and $\vec{z} \in W^{\perp}$, then $||\vec{x}||^2 = ||\vec{y}||^2 + ||\vec{z}||^2$.

(13) T F If the columns of $m \times n$ U are orthonormal, then the rows of U are also orthonormal.

(14) T F Suppose $\vec{u}, \vec{v} \in R^{10}$ form a basis for W, so $\dim W = 2$. Then the set of all vectors that are orthogonal to W is a subspace of R^8.

(15) T F Suppose $\hat{\vec{b}} = Proj_W \vec{b}$. Then $\vec{b} = Proj_W \hat{\vec{b}}$.

10 points 1. We'll find a 2×2 matrix by looking at what the matrix and its inverse do to vectors.

Given initial vector $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $\vec{x}_{k+1} = \boldsymbol{A}\vec{x}_k$, we have (approximately)

k	0	1	$\cdots$	10	11
$\vec{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 14 \\ 18 \end{bmatrix}$	$\cdots$	$\begin{bmatrix} 1.5 \times 10^9 \\ 2.0 \times 10^9 \end{bmatrix}$	$\begin{bmatrix} 1.5 \times 10^{10} \\ 2.0 \times 10^{10} \end{bmatrix}$

and (now using $\vec{x}_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\boldsymbol{A^{-1}}$) where $\vec{x}_{k+1} = \boldsymbol{A^{-1}}\vec{x}_k$ we have (approximately)

k	0	1	$\cdots$	51	52
$\vec{x}_k$	$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}$	$\cdots$	$\begin{bmatrix} 4.44 \times 10^{-16} \\ 8.88 \times 10^{-16} \end{bmatrix}$	$\begin{bmatrix} 2.22 \times 10^{-16} \\ 4.44 \times 10^{-16} \end{bmatrix}$

Find A. Hint: first, estimate the eigenvalues and eigenvectors, and then use these to write A as the produce of three matrices. Note: you are welcome to leave A written as the product of three matrices and NOT actually do the computation to find A.

26 points 2. Suppose $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$.

/8 (a) Find the least squares solution $\hat{\vec{x}}$ to $A\vec{x} = \vec{b}$. Show appropriate work.

/2 (b) Find $\hat{\vec{b}}$, the projection of $\vec{b}$ onto $Col\,A$. That is, compute $\hat{\vec{b}} = A\hat{\vec{x}}$ using the $\hat{\vec{x}}$ that you found in (a).

/5 (c) Next, use the Gram-Schmidt Process to find an orthogonal basis (call these vectors $\vec{v}_1, \vec{v}_2$) for $Col\,A$. That is, find $\vec{v}_1, \vec{v}_2$ from the two columns of A. Show appropriate work.

Problem 2 is continued on this page

/5 (d) Find the projection $Proj_{Span\{\vec{v}_1,\vec{v}_2\}}\vec{u}$ for the $\vec{v}_1, \vec{v}_2$ that you found in (c).

/6 (e) Finally, using the work that you did in (c), find the QR factorization of A, where Q has orthonormal columns and R is upper triangular.

20 points 3. Suppose $\vec{x} = \begin{bmatrix} 1 \\ -4 \\ -3 \\ 5 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, and $W = span\{\vec{u}_1, \vec{u}_2\}$.

/7 (a) Find vectors $\vec{y} \in W$ and $\vec{z} \in W^T$ so that $\vec{x} = \vec{y} + \vec{z}$. (Notice that $\vec{u}_1 \perp \vec{u}_2$.)

/1 (b) Briefly explain why $\dim W^{\perp} = 2$.

/6 (c) Find two vectors that form a basis for $W^{\perp}$. (You might want to first double check that your two vectors are orthogonal to $\vec{v}_1, \vec{v}_2$.)

/6 (d) Show that the vector $\vec{z}$ that you found in (a) is indeed in $W^{\perp}$. Do this by finding how you would build (as a linear combination) $\vec{z}$ from the two vectors you found in (c).

For the final two problems, $A = PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -3 \\ -6 & 0 \end{bmatrix}$.

8 points 4. The sizes of two competing populations $\vec{x}$ change according to $\vec{x}_{k+1} = A\vec{x}_k$.

Where $\vec{x}_0 = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$, find $\vec{x}_1$. In general, what is $\vec{x}_k$?

6 points 5. The position $\vec{x}$ of a particle in a planar force field satisfies the equation $\frac{d\vec{x}}{dt} = A\vec{x}$.

Where $\vec{x}(0) = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$, find the position of the particle at time t.