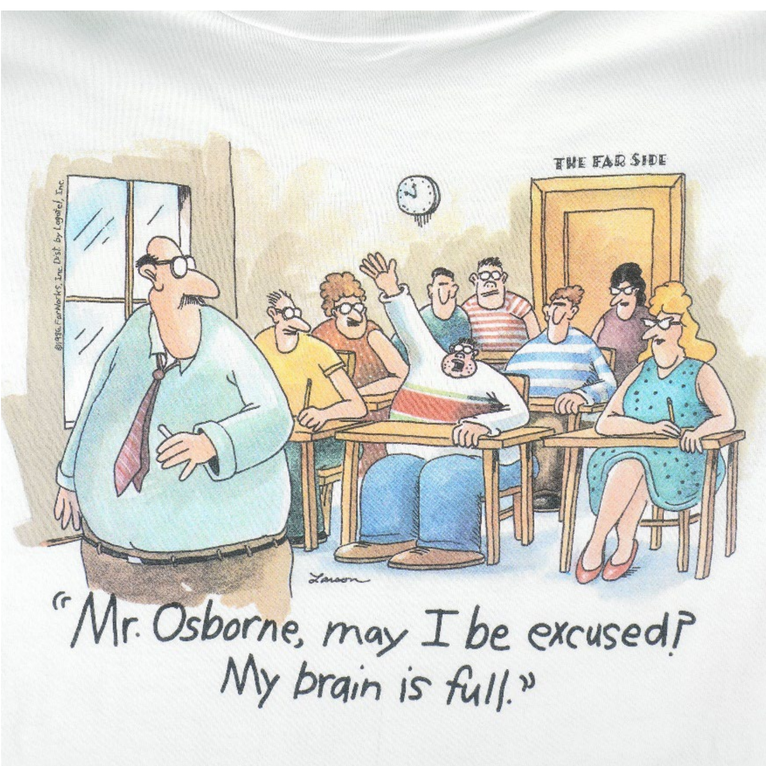
Math 260 Fall 2024 **Exam 2** **March 15, 2024**

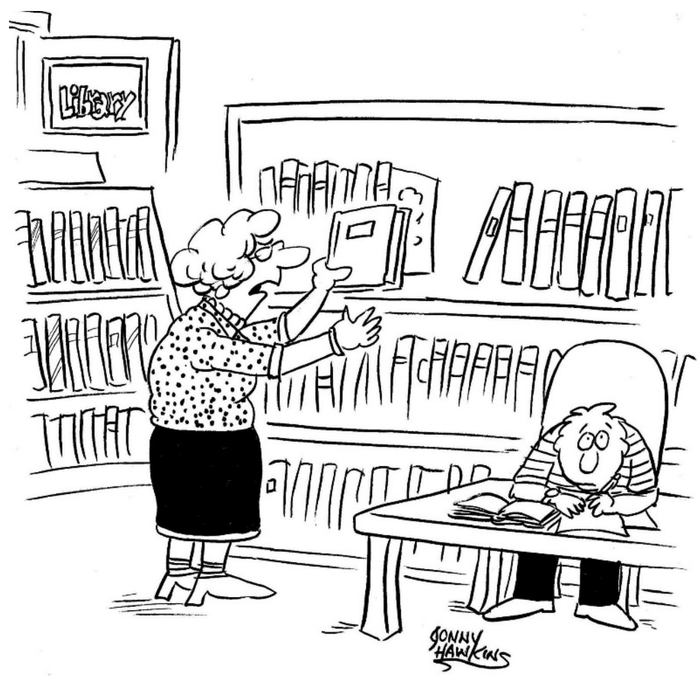
Name: Solutions

Problem	T/F	1 / 2	3	4 / 5	Total
Possible	50	15	15	20	100
Received					

YOU MAY USE A 3 X 5 CARD OF NOTES.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.





"Who is putting all the Math books in the Horror section?"

50 points T/F. Answer the following 25 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may be true sometimes or under some circumstances, but not always or not necessarily. No explanation is necessary whether true or false.

(1) T F If 4×5 matrix A has two pivot columns, then $Col\ A = R^2$.

A 2-dim. subspace of $\mathbb{R}^5$, but not $\mathbb{R}^2$.

(2) T F If $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$ where $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A and $\lambda_1 \neq \lambda_2$, then it must be that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

(3) T F For matrix A, if $\vec{v}_1$ is an eigenvector corresponding to eigenvalue λ_1 and $\vec{v}_2$ is an eigenvector corresponding to eigenvalue λ_2, then $\vec{v}_1 - \vec{v}_2$ is an eigenvector corresponding to eigenvalue $\lambda_1 - \lambda_2$.

(4) T F For 3×5 matrix A, $3 \leq rank\ A \leq 5$. rank $A \leq \min(3, 5)$

(5) T F If a $\boxed{5 \times 5}$ matrix A has rank 4, then $A\vec{x} = \vec{b}$ could have a unique solution for some right hand side $\vec{b}$.

square

$\Rightarrow$ A is "bad" $\Rightarrow$ free variable

$\Rightarrow$ if there is a sol'n., then there are ∞ solns.

(6) T F If a 5×5 matrix A has rank 4, then $A\vec{x} = \vec{b}$ will have no solution for some right hand side $\vec{b}$ and an infinite number of solutions for some other $\vec{b}$.

(7) T F If a 5×5 matrix A has rank 4, then $A\vec{x} = \vec{b}$ will have an infinite number of solutions for some right hand side $\vec{b}$, no solution for some other right hand side $\vec{b}$, and a unique solution for yet another right hand side $\vec{b}$.

(8) T F The area of the parallelogram with vertices of the origin, (1,3) and (2,1) is 5.

Abs. value of $\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$.

(9) T F For any $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in R^5, $span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a subspace of R^5.

(10) T F $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ is in both the column space and the null space of $\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$.

(11) T F $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 6 & 1 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 8 & 2 \end{bmatrix}^{-1}$.

Re-order e-vectors/values in $A = PDP^{-1}$

(12) T F For a 3×5 matrix, the dimensions of the row space and column space will be different.

Always = # pivots

(13) T F If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for R^3, then the matrix $[\,\vec{v}_1\ \ \vec{v}_2\ \ \vec{v}_3\,]$ is invertible.

$\implies$ lin. ind. $\Rightarrow$

(14) T F If $\det A = 2$ and $\det B = -3$, then $\det(A+B) = -1$.

(15) T F $\det(A^T) = -\det A$.

(16) T F If $A\vec{x} = \vec{b}$, then $\vec{x}$ is in the column space of A.

$\cancel{\vec{x}}$ $\vec{b}$

(17) T F $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of matrix $\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, no matter what a, b and c are.

$= (a+b+c)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(18) T F The determinant of $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 0 \end{bmatrix}\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{\text{det. is } 0}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 0 \end{bmatrix}^{-1}$ is 0.

and $\det ABC = \det A \cdot \det B \cdot \det C$.

(19) T F If the determinant of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ were 5 (it's actually not), then the determinant of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}$ would also be 5.

Row 3 = Row 1 + Row 2 doesn't change det.

(20) T F If the determinant of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ were 5 (it's actually not), then the determinant of $\begin{bmatrix} 0 & 0 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ would also be 5.

Swapping rows changes sign +/- of det.

(21) T F If the determinant of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ were 5 (it's actually not), then the determinant of $\begin{bmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 0 & 0 & 0 \end{bmatrix}$ would be $\cancel{5}0$.

$5 \cdot 10 \cdot 10$.

For the final four problems, V is a vector space, and vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ come from V.

(22) T F If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for V, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent.

(23) T F If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for V.

(24) T F If $span\,\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = V$, then some subset of $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for V.

(25) T F If $\dim V = n$ and $span\,\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = V$, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent.

So $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for V, so

5 points 1. If a 5 $\times$ 9 matrix A has rank 3, then: dim $Row\ A$ = 3

$rank\ A^T$ = 3

dim $Nul\ A$ = $9 - 3 = 6$

If a 9 $\times$ 5 matrix A has rank 3, then: dim $Nul\ A$ = $5 - 3 = 2$

If a 5 $\times$ 5 matrix A has rank 3, then: dim $Nul\ A$ = $5 - 3 = 2$

10 points 2. Consider the probability matrix $A = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$, which has eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and corresponding eigenvalues $-1/6$ and 1.

First, suppose that $\vec{x}_0 = \begin{bmatrix} 500 \\ 500 \end{bmatrix}$. Find $\vec{x}_0$ as a linear combination of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. That is find $[\vec{x}_0]_B$ where $B = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$. Don't just guess—show your work.

$$\begin{bmatrix} 500 \\ 500 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 500 \\ 500 \end{bmatrix} = \frac{1}{1 \cdot 4 - (-1)(3)} \begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 500 \\ 500 \end{bmatrix}$$

$$= \begin{bmatrix} 500/7 \\ 1000/7 \end{bmatrix}. \quad \text{So } \begin{bmatrix} 500 \\ 500 \end{bmatrix} = \frac{500}{7}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1000}{7}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Second, find $\vec{x}_\infty = A^\infty \vec{x}_0$ $\left(\text{that is, find } \lim_{k\to\infty} A^k \vec{x}_0\right)$. Again, show your work.

$$\text{So } A^k \vec{x}_0 = A^k \left(\begin{bmatrix} 500/7 \\ -500/7 \end{bmatrix} + \begin{bmatrix} 3000/7 \\ 4000/7 \end{bmatrix} \right)$$

$$= \left(-\frac{1}{6}\right)^k \begin{bmatrix} 500/7 \\ -500/7 \end{bmatrix} + 1^k \begin{bmatrix} 3000/7 \\ 4000/7 \end{bmatrix}$$

which $\longrightarrow \begin{bmatrix} 3000/7 \\ 4000/7 \end{bmatrix}$ as $k \to \infty$.

Notice: sum is 1000 ($= 500 + 500$) and the 3 : 4 ratio, as in the e-vector for e-value 1.

11 points 3. Consider matrix A which is row equivalent to matrix B:

$$A = \begin{bmatrix} 1 & -5 & 1 & 2 & -1 & 1 \\ 1 & -5 & 2 & 0 & -3 & 3 \\ 1 & -5 & 3 & -2 & 3 & 13 \\ 1 & -5 & 4 & -4 & 5 & 19 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -5 & 0 & 4 & 0 & -2 \\ 0 & 0 & 1 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

First, even though there are six columns, based on the dimensions of A, how do you know that $\dim Col\, A \leq 4$? dim Col A = dim Row A ≤ # rows

Find the following:

$rank\ A =$ 3   3 pivots in cols. 1, 3, 5.

A basis for $Col\ A$: Cols. 1, 3, 5 of A

A basis for $Row\ A$: Rows 1, 2, 3 of A or B

A basis for $Nul\ A$:

Free variables: x_2, x_4, x_6

$$\begin{aligned} x_1 - 5x_2 + 4x_4 - 2x_6 &= 0 \\ x_3 - 2x_4 + 4x_6 &= 0 \\ x_5 + x_6 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 2 \\ 0 \\ -4 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Col. 2

9 points 4. Use Cramer's Rule to find x_2 in the linear system

$$\begin{aligned} -x_1 + 2x_2 + 3x_3 &= 0 \\ 3x_2 + 5x_3 &= -2 \\ x_1 + 4x_2 &= 0 \end{aligned}.$$

$$x_2 = \frac{\begin{vmatrix} -1 & 0 & 3 \\ 0 & -2 & 5 \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 1 & 4 & 0 \end{vmatrix}} = \frac{-2\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix}}{-1\begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} = \cdots = \frac{6}{21}$$

15 points 5. Find the two eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 2 & 2 \\ -3 & 7 \end{bmatrix}$.

Hint in factoring: $4 \cdot 5 = 20$

$$\begin{vmatrix} 2-\lambda & 2 \\ -3 & 7-\lambda \end{vmatrix} = (2-\lambda)(7-\lambda) + 6 = \lambda^2 - 9\lambda + 20$$
$$= (\lambda - 4)(\lambda - 5) = 0$$
$$\Rightarrow \lambda = 4, 5.$$

$\underline{\lambda = 4}$: $A\vec{x} = 4\vec{x}$, i.e. $(A - 4I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= x_2 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 5}$

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ -3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 &= \tfrac{2}{3}x_2 \\ x_2 &= x_2 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Find matrices X and D so that $A = XDX^{-1}$.

$$A = \underbrace{\begin{bmatrix} 1 & \frac{2}{3} \\ 1 & 1 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & \frac{2}{3} \\ 1 & 1 \end{bmatrix}^{-1}}_{X^{-1}}$$

How do you know that X is invertible?

It's cols. are lin. ind. (which was guaranteed since they correspond to two different e-values).

BTW, this has nothing to do with A's e-values being 0 or not.