Math 260 Spring 2024 | Exam 1 | February 9, 2024

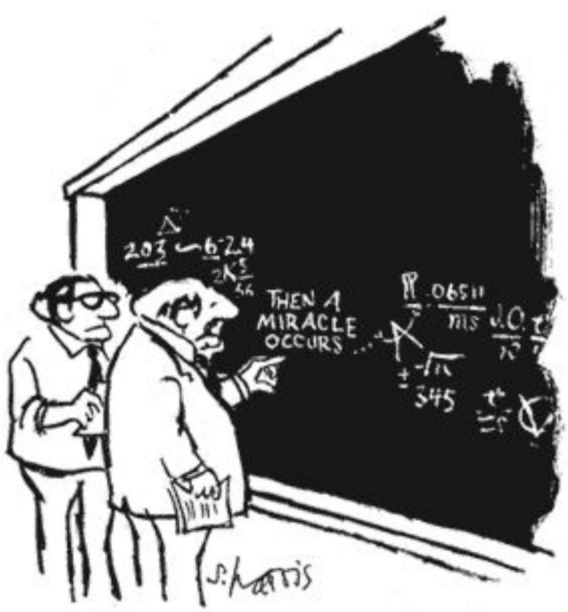
Name: Solutions

Problem	T/F	1 / 2	3 / 4 / 5	6 / 7	8	Total
Possible	40	15	20	15	10	100
Received						

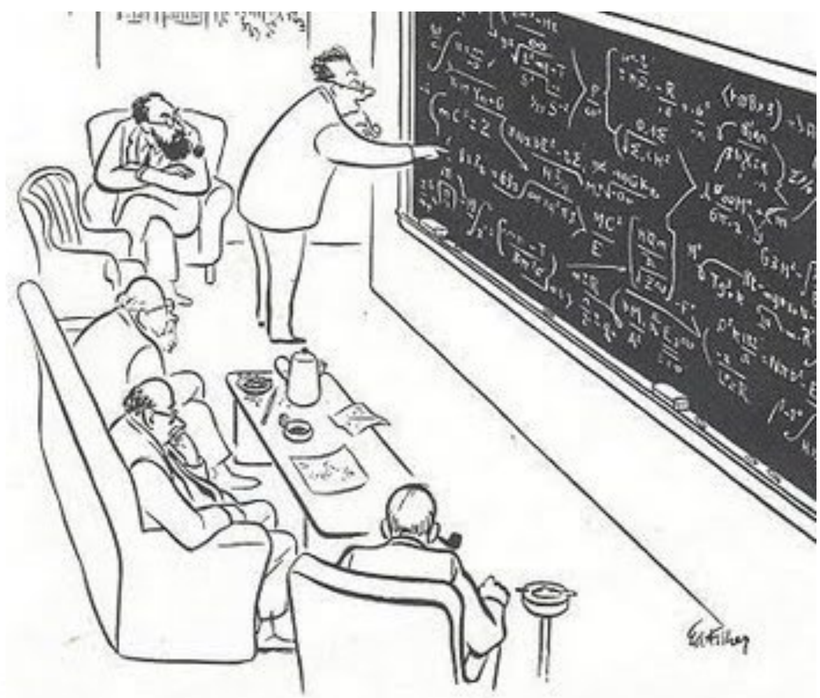
Be sure to show all pertinent work to receive full credit.

You may use a 3” x 5” card of handwritten notes, both sides.

You will not use a calculator on this exam.



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."



“Say, I think I see where we went off. Isn’t eight times seven fifty-six?”

40 points T/F. Answer the following 20 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may sometimes be true, but not always or not necessarily. *No work or explanation or justification is needed to be shown for these questions.*

True False If the columns of 4×5 (4 rows, 5 columns) A span R^4, then the columns are linearly independent. 5 vectors from $\mathbb{R}^4$ can never be lin. ind.

True False If $Span\{\vec{v}_1, \vec{v}_2\} = Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent.

Not neccesarily — could be.

True False In R^3 if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then $Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a plane in R^3. Might be, might not. Example: $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}3\\3\\3\end{bmatrix}\right\}$

True False $\begin{bmatrix}1.1\\-\sqrt{\pi}\end{bmatrix}$ can be written as a linear combination of $\left\{\begin{bmatrix}1\\3.14\end{bmatrix}, \begin{bmatrix}1/2\\1/3\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$.

Since span $\mathbb{R}^2$.

True False For 3×3 matrix A, the first column of A^{-1} is the same as the solution $\vec{x}$ to the problem $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

True False If the columns of $n \times n$ matrix A are linearly <u>independent</u>, then $A\vec{x} = \vec{b}$ <u>will</u> have a solution, no matter what $\vec{b}$ is. See Inv. Matrix Thm.

True False If the columns of $n \times n$ matrix A are linearly <u>dependent</u>, then $A\vec{x} = \vec{b}$ will <u>not</u> have a solution, no matter what $\vec{b}$ is.

Maybe, maybe not. Example: $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$.

True False For $A\vec{x} = \vec{b}$, it is possible that $A\vec{x} = \vec{b}_1$ has <u>exactly one</u> solution for some $\vec{b}_1$ while $A\vec{x} = \vec{b}_2$ has <u>no</u> solution for some other $\vec{b}_2$.

Notice A is not necc. square. Example: $A = \begin{bmatrix}1 & 2\\1 & 3\\1 & 4\end{bmatrix}$, $\vec{b}_1 = \begin{bmatrix}3\\4\\5\end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix}3\\4\\6\end{bmatrix}$

True False If $T(x_1, x_2) = (2x_1 + 3x_2, 4x_1 + 5x_2)$, then $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$.

True False If $n \times n$ matrices A, B, C and D are invertible, then $(ABAB)^{-1} = A^{-1}B^{-1}A^{-1}B^{-1}$.

True False $\begin{bmatrix} a^2 \\ b \end{bmatrix}, \begin{bmatrix} a \\ 1 \end{bmatrix}$ are linearly dependent if $a = b$, and linearly independent otherwise.

$\frac{a^2}{a} = \frac{b}{1} \Rightarrow a^2 = ab \Rightarrow a(a-b) = 0$
$\Rightarrow a = 0, a = b.$

True False If $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, then $A^T = A$.

True False If $A = \begin{bmatrix} 1 & c & 2 \\ 1 & 0 & 2 \\ 1 & c & 2 \end{bmatrix}$, where c is some non-zero constant, then $A^{-1} = A$.

Cols. lin. dep., so no inverse.

True False If $A = \begin{bmatrix} 1 & c & 2 \\ 1 & 0 & 2 \\ 1 & c & 2 \end{bmatrix}$, then $A\vec{x} = \vec{b}$ must have more than one solution for some $\vec{b} \neq \vec{0}$.

Example: $\vec{b}$ is a multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

True False The columns of a 3×2 matrix could be linearly dependent.

True False The columns of a 3×2 matrix could span R^2.

Cols. come from $\mathbb{R}^3$, not $\mathbb{R}^2$.

True False The columns of a 2×3 matrix must span R^2.

True False If A is a 2×3 matrix and B is a 3×4 matrix, then $(AB)^T(AB)$ is a 4×4 matrix.

True False The problem of solving for x_1 and x_2 in the system of equations

$$2x_1 + 3x_2 = 4$$
$$5x_1 + 6x_2 = 7$$

is equivalent of solving for x_1 and x_2 in the vector equation

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

True False Where A and B are matrices, any vector that can be written as $AB\vec{x}$ for some vector $\vec{x}$ is a linear combination of the columns of A.

For any $\vec{v}$ (including $\vec{v} = B\vec{x}$), $A\vec{v}$ is a lin. comb. of cols. of A.

9 points 1. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

where the right-hand block is A^{-1}.

Notice that just as A has a certain pattern, A^{-1} also has a pattern.

6 points 2. Suppose $A\vec{x}_1 = \vec{b}$ and $A\vec{x}_2 = \vec{b}$ for $\vec{x}_1 \neq \vec{x}_2$. Show that $A\vec{x} = \vec{0}$ has a non-trivial (i.e. non-zero) solution $\vec{x}$, and show that $A\vec{x} = \vec{b}$ for the same right-hand side $\vec{b}$ has an infinite number of solutions.

Where $\vec{x}_h = \vec{x}_1 - \vec{x}_2$, which $\neq \vec{0}$, since $\vec{x}_1 \neq \vec{x}_2$,

then $A\vec{x}_h = A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}$.

Then for any c (and infinite number of values),

$$\begin{aligned} A(\vec{x}_1 + c\vec{x}_h) &= A\vec{x}_1 + cA\vec{x}_h \\ &= \vec{b} + c\vec{0} \\ &= \vec{b}. \end{aligned}$$

or $\vec{x}_2$

BTW, A is not neccesarily square.

6 points 3. Suppose for $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

you have $\underbrace{E_3E_2E_1}_{A^{-1}}A = I$. Find A.

$$A = (A^{-1})^{-1} = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \dots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

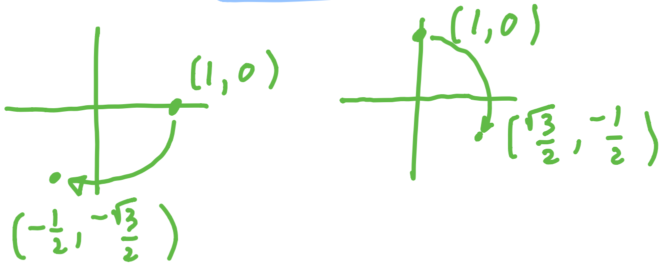
6 points 4. Find the inverse of $\left[\begin{array}{cc|cc} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{array}\right]$. Hint: think partitioned/block matrices.

Show your work.

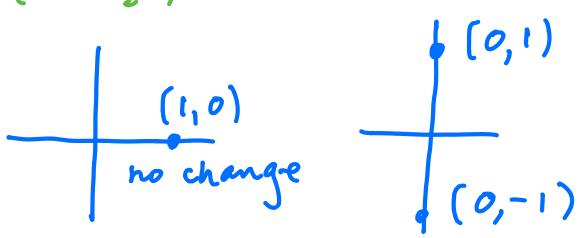
$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}^{-1} = \frac{1}{1\cdot 1 - (-2)(-2)}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

so the inverse is $\begin{bmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & \end{bmatrix}$

8 points 5. Find the 2×2 matrix which first rotates clockwise by 120 degrees, and then reflects across the x-axis.



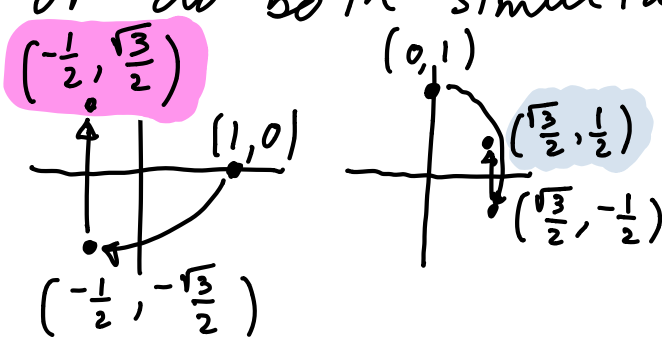
$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

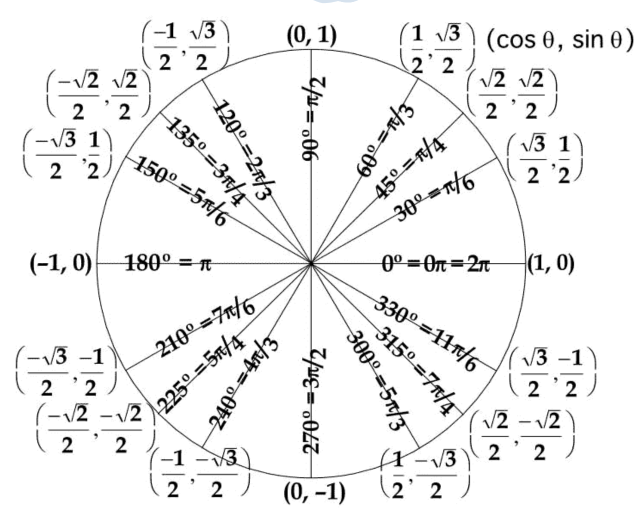


$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

or do both simultaneously:





5 points 6. Suppose that you have some nickels (5 cents) and dimes (10 cents) and you want \$4.00 using exactly 46 coins.

Set up a system of two equations and two unknowns which corresponds to this information. (Let n be the number nickels and d be the number of dimes.)

$$n + d = 46$$
$$5n + 10d = 400$$

Before solving for n and d, why do you expect there to be exactly one solution?

equations = # unknowns

Now find the needed number of nickels and dimes by solving this system using row reduction or using an appropriate matrix inverse.

$$\left[\begin{array}{cc|c} 1 & 1 & 46 \\ 5 & 10 & 400 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 46 \\ 0 & 5 & 170 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 46 \\ 0 & 1 & 34 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 12 \\ 0 & 1 & 34 \end{array}\right]$$

so $n = 12$, $d = 34$.

10 points 7. A company produces two items, but some of each product is consumed during the production process as described by the consumption matrix

$$C = \begin{bmatrix} .6 & .2 \\ 0 & .5 \end{bmatrix}$$

How much should they produce if they want to end up with 10 units of each product? What is one thing about your solution that makes it seem reasonable

Note for this problem that $(.5)(.4) = .2$ and that $\frac{.5}{.2} = \frac{5}{2}$ and $\frac{.4}{.2} = 2$.

$$(I - C)^{-1} = \begin{bmatrix} .4 & -.2 \\ 0 & .5 \end{bmatrix}^{-1} = \frac{1}{\underbrace{(.4)(.5)}_{.2} - 0(-.2)} \begin{bmatrix} .5 & .2 \\ 0 & .4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & 1 \\ 0 & 2 \end{bmatrix}$$

So produce $\begin{bmatrix} \frac{5}{2} & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 35 \\ 20 \end{bmatrix}$, which $> \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

How much more of each product needs to be produced if demand for Product Two were to increase by 1 unit (and demand for Product One does not change)?

$\begin{bmatrix} \frac{5}{2} & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 11 \end{bmatrix} = \begin{bmatrix} 36 \\ 22 \end{bmatrix}$ which is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ more, column two of $(I - C)^{-1}$

10 points 8. Consider $A = \begin{bmatrix} 6 & -7 & -27 & -3 & 17 \\ -6 & 9 & 33 & 4 & -19 \\ -1 & 2 & 7 & 1 & -4 \end{bmatrix}$ which is the coefficient matrix for the system of 3 equations and 5 unknowns $A\vec{x} = \vec{b}$ with augmented matrix

$$\left[\begin{array}{ccccc|c} 6 & -7 & -27 & -3 & 17 & 48 \\ -6 & 9 & 33 & 4 & -19 & -53 \\ -1 & 2 & 7 & 1 & -4 & -10 \end{array}\right].$$

I've done all of the work to find that

$$\left[\begin{array}{ccccc|c} 6 & -7 & -27 & -3 & 17 & 48 \\ -6 & 9 & 33 & 4 & -19 & -53 \\ -1 & 2 & 7 & 1 & -4 & -10 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & -4 & 1 \end{array}\right].$$

/6 Give the general solution to the above problem $A\vec{x} = \vec{b}$ and give two specific solutions. Which part of your solution is the solution to the homogeneous problem $A\vec{x} = \vec{0}$?

$$\begin{aligned} x_1 \quad - x_3 \quad + 2x_5 &= 5 \\ x_2 + 3x_3 \quad + x_5 &= -3 \\ x_4 - 4x_5 &= 1 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

General solution.

Specific solutions: pick specific values for x_3, x_5.

/2 What is it about the reduced row echelon matrix $\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & -4 & 1 \end{array}\right]$ that tells you that there is a solution to $A\vec{x} = \vec{b}$ no matter what the right hand side $\vec{b}$ is?

Pivot in every row.

/2 Which columns of $\begin{bmatrix} 6 & -7 & -27 & -3 & 17 \\ -6 & 9 & 33 & 4 & -19 \\ -1 & 2 & 7 & 1 & -4 \end{bmatrix}$ form a linearly independent set? Just circle the columns in this matrix.

The pivot columns.

/2 Extra-credit: write each of the dependent columns of A as linear combinations of the independent ones.

$$\begin{bmatrix} -27 \\ 33 \\ 7 \end{bmatrix} = -1 \begin{bmatrix} 6 \\ -6 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -7 \\ 9 \\ 2 \end{bmatrix}; \quad \begin{bmatrix} 17 \\ -19 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 6 \\ -6 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} -7 \\ 9 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

Invertible Matrix Theorem for $n \times n$ matrix A

a. A is invertible.

b. A is row equivalent to I.

c. A has n pivot positions.

d. A**x** = **0** has only trivial solution.

e. Columns of A lin. independent.

f. Linear transf. $\mathbf{x} \to A\mathbf{x}$ 1-to-1.

g. A**x** = **b** has at least one solution for each **b**.

h. Columns of A span $\mathbf{R}^n$.

i. Linear transf. $\mathbf{x} \to A\mathbf{x}$ onto.

j. There is C such that CA = I.

k. There is D such that AD = I.

l. A^T is invertible.

m. Columns of A form basis for $\mathbf{R}^n$.

n. Column space of A is $\mathbf{R}^n$.

o. dim Col A = n, *i.e.* dimension of column space of A is n.

p. rank A = n, *i.e.* rank of A is n.

q. Nul A = {**0**}, *i.e.* nullspace of A is {**0**}.

r. dim Nul A = 0, the dimension of the null space of A is 0.

s. A has n nonzero eigenvalues, *i.e.* 0 is not an eigenvalue of A.

t. det A $\neq$ 0.

u. $(\text{Col } A)^\perp = \{\mathbf{0}\}$, *i.e.* orthogonal complement of column space of A is {**0**}.

v. $(\text{Nul } A)^\perp = \mathbf{R}^n$, *i.e.* orthogonal complement of null space of A is $\mathbf{R}^n$.

w. Row A = $\mathbf{R}^n$, row space of A is $\mathbf{R}^n$.