**Math 260 Fall 2022** **Exam 3** **November 21, 2022**

Name: Solutions

| Problem | T/F | 1 | 2 | 3 / 4 | Total |
|---|---|---|---|---|---|
| Possible | 40 | 32 | 12 | 16 | 100 |
| Received | | | | | |

**DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.**

**You may use a 3 × 5 card of notes,**

**FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.**







40 points T/F. Answer the following 20 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may be true sometimes or under some circumstances, but not always or not necessarily. No explanation is necessary whether true or false.

True False If $W = span\{\vec{v}_1, \vec{v}_2\}$, then $Proj_W\vec{u} = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$.

Only if $\vec{v}_1 \perp \vec{v}_2$

True False $\vec{u} - Proj_W\vec{u}$ is parallel to $W$.

orthogonal

True False $A\vec{x} = \vec{b}$ always has a ~~unique~~ least squares solution.

True False If $\hat{\vec{x}}$ is the least squares solution to $A\vec{x} = \vec{b}$, then for any other $\vec{x}$ we have $\|\vec{b} - A\hat{\vec{x}}\| \leq \|\vec{b} - A\vec{x}\|$.

True False The angle between vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ -5 \\ 6 \end{bmatrix}$ is $90°$.

Since $\vec{u} \cdot \vec{v} = 0$

True False The distance between vectors $\vec{u}$ and $\vec{v}$ is $\|\underbrace{\vec{u} + \vec{v}}_{\vec{u} - \vec{v}}\|$.

True False For a square matrix $A$, vectors in $Col\ A$ are orthogonal to vectors in $Nul\ A^T$.

True False If $\|\vec{u} + \vec{v}\| < \|\vec{u}\| + \|\vec{v}\|$, then $\vec{u} \cdot \vec{v} \neq 0$.

Always $\leq$. Only $=$ if $\vec{u}$ and $\vec{v}$ are parallel, so $<$ for any non-parallel $\vec{u}$, $\vec{v}$.

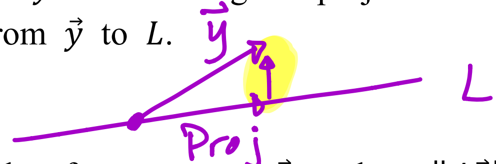
True False A linearly independent set of non-zero vectors must be an orthogonal set.

True False An orthogonal set of non-zero vectors is linearly independent.

True False $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for $R^4$.

Need $\geq 4$ vectors for $\mathbb{R}^4$ basis.

True False If $L$ is a line through the origin and if $\hat{\vec{y}}$ is the orthogonal projection of $\vec{y}$ onto $L$, then $\|\vec{y} - \hat{\vec{y}}\|$ gives the distance from $\vec{y}$ to $L$.




True False If the columns of $A$ are orthonormal, then for any vector $\vec{x}$ we have $\|A\vec{x}\| = \|\vec{x}\|$.

True False If $Proj_{\vec{v}}\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $Proj_{3\vec{v}}\vec{u} = \cancel{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}$. is still $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

True False If $Proj_{\vec{v}}\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $Proj_{\vec{v}}3\vec{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

True False If $U$ and $V$ are orthogonal matrices (if $U^TU = I$ and $V^TV = I$), then their product $UV$ is orthogonal.

True False Given a linearly independent set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, dividing each by its own length $\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \ldots, \frac{\vec{v}_n}{\|\vec{v}_n\|}\}$ will result in an orthonormal set of vectors.

These will be length 1, but not neccesarily orthogonal.

True False If $\lambda_1$ and $\lambda_2$ are eigenvalues of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then $|\lambda_1| = 1$ and $|\lambda_2| = 1$.

$A^TA = I$, so all e-values are magnitude 1.

Or compute e-values of $\cos\theta + i\sin\theta$, thus

True False If the columns of $U$ form an orthonormal basis for $R^n$, then for $\vec{y} \in R^n$, $UU^T\vec{y} = \vec{y}$.

True False The projection of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ onto $span\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}\}$ is $\begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix}$.

↳ is itself, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, since span $\mathbb{R}^2$.

32 points 1. We are interested in finding the solution to

$$\begin{aligned} x + 3y &= 0 \\ x + 2y &= -1 \\ x + y &= 4 \end{aligned}$$

/7 With more equations than unknowns, likely this system does not have an exact solution. So let's do the best we can. First (and being <u>careful with your arithmetic</u>) find the least squares solution $\hat{\vec{x}}$ to $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}.$$

$A^TA = \cdots = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$, inverse is $\frac{1}{3\cdot 14 - 6\cdot 6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$

$A^T\vec{b} = \cdots = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

So $\hat{\vec{x}} = \frac{1}{6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

/2 Compute $A\hat{\vec{x}}$ using the above $\hat{\vec{x}}$ you just found. (Recall that $\hat{\vec{b}} = A\hat{\vec{x}} = Proj_{\,Col\,A}\,\vec{b}$ is the closest we can get to building vector $\vec{b}$ using the columns of $A$.)

$\hat{\vec{b}} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$.

/2 Find $\vec{b} - \hat{\vec{b}}$, and confirm that $\vec{b} - \hat{\vec{b}} \perp$ the columns of $A$.

$\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, whose $\cdot$ product with $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is 0.

Problem 1 continued (careful in doing your arithmetic!)

/6 We still want to find $Proj_{Col\,A}\,\vec{b}$, where $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$. But this time we'll do it a bit differently. First, use the Gram-Schmidt Process to construct two orthogonal vectors whose span is the same as the two columns of matrix $A$. That is, where $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, find $\vec{v}_1$ and $\vec{v}_2$ so that $span\{\vec{v}_1, \vec{v}_2\} = span\{\vec{u}_1, \vec{u}_2\}$, but where $\vec{v}_1 \perp \vec{v}_2$.

$$\vec{v}_1 = \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \frac{6}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(Notice: $\vec{v}_2 = \vec{u}_2 - 2\vec{v}_1$, so $2\vec{v}_1 + \vec{v}_2 = \vec{u}_2$.)

/7 Of course your $\vec{v}_1$ and $\vec{v}_2$ should be orthogonal. Using this fact, find $Proj_{\vec{v}_1,\vec{v}_2}\,\vec{b}$ (or if you prefer the alternate notation, find $Proj_W\,\vec{b}$, where $W = span\{\vec{v}_1, \vec{v}_2\}$).

$$\hat{\vec{b}} = \frac{\vec{b} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{b} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \frac{3}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{-4}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

Same as on previous page.

/8 Using your work above, find the $QR$-factorization of $A$ where $Q$ has orthogonal columns and $R$ is upper triangular. (+1 point extra credit if you find $Q$ and $R$ so that the columns of $Q$ are ortho<u>normal</u>).

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{R} = \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}}_{A}$$

Or find $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$, so $Q^TQ = I$,

and $QR = A \Rightarrow \underline{Q^TQR = Q^TA} = \begin{bmatrix} \sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix} = A$$

12 points 2. We'll determine what a $2 \times 2$ matrix $A$ does to a vector by examining the long term behavior of multiplying some vector by $\boldsymbol{A}$ and by $\boldsymbol{A^{-1}}$.

Given initial vector $\vec{x}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, where $\vec{x}_{k+1} = \boldsymbol{A}\, \vec{x}_k$, we have (approximately)

e-vector e-value

| $k$ | 0 | 1 | $\cdots$ | 10 | 11 |
|---|---|---|---|---|---|
| $\vec{x}_k$ | $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ | $\begin{bmatrix} 9 \\ 3 \end{bmatrix}$ | $\cdots$ | $\begin{bmatrix} 3 \times 10^5 \\ 3 \times 10^5 \end{bmatrix}$ | $\begin{bmatrix} 9 \times 10^5 \\ 9 \times 10^5 \end{bmatrix}$ |

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_1 = 3$

and (now using $\vec{x}_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$) where $\vec{x}_{k+1} = \boldsymbol{A^{-1}}\, \vec{x}_k$ we have (approximately)

| $k$ | 0 | 1 | $\cdots$ | 27 | 28 |
|---|---|---|---|---|---|
| $\vec{x}_k$ | $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ | $\begin{bmatrix} -0.17 \\ 1.33 \end{bmatrix}$ | $\cdots$ | $\begin{bmatrix} 2 \times 10^{-8} \\ 4 \times 10^{-8} \end{bmatrix}$ | $\begin{bmatrix} 1 \times 10^{-8} \\ 2 \times 10^{-8} \end{bmatrix}$ |

$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\lambda_2 = 2$

Find $A^3\vec{x}$ where $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Hint: estimate the eigenvalues and eigenvectors from the above information, find $\vec{x}$ as a linear combination of those two eigenvectors, and then compute $A^3\vec{x}$.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \cdots = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Then $A^3\vec{x} = A^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A^3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2^3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 35 \\ 43 \end{bmatrix}$.

Or if you like doing more work :)

$A^3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3^3 & 0 \\ 0 & 2^3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \cdots = \begin{bmatrix} 46 & -19 \\ 38 & -11 \end{bmatrix}$, so $A^3\vec{x} =$

Or $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \cdots = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$,

then multiply $\vec{x}$ by $A$ 3 times to get

/8 3. Given functions $f(t) = t$ and $g(t) = t^2$, use the Gram-Schmidt Process to find two functions that are orthogonal under the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$.

Keep $t$. Then use GS:

$$g - \frac{\langle g, f \rangle}{\langle f, f \rangle} f = t^2 - \frac{\int_0^1 t^2 \cdot t \, dt}{\int_0^1 t \cdot t \, dt} t$$

$$\ldots = t^2 - \frac{1/4}{1/3} t = t^2 - \frac{3}{4} t.$$

You could check that $\langle t, t^2 - \frac{3}{4}t \rangle = 0$.

/8 4. Suppose that for two populations (let's say Owls $O$ and Rats $R$), in month $k$ their populations are $O_k$ and $R_k$ where

$$O_{k+1} = 1.42\, O_k + 0.02\, R_k$$
$$R_{k+1} = 0.08\, O_k + 1.48\, R_k$$

with initial populations of $\begin{bmatrix} O_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$. The eigenvectors of $\begin{bmatrix} 1.42 & 0.02 \\ 0.08 & 1.48 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ with corresponding eigenvalues of 1.4 and 1.5 respectively.

Find a general expression/formula for $\vec{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$.

$$\vec{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix} = c_1 (1.4)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 (1.5)^k \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

where $\begin{bmatrix} 7 \\ 3 \end{bmatrix} = \vec{x}_0 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \Rightarrow$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \cdots = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

/2 Extra credit. The above is the discrete version of this problem. What would the continuous version of this problem be and what would the corresponding solution be?

$$\begin{bmatrix} \frac{dO}{dt} \\ \frac{dR}{dt} \end{bmatrix} = \begin{bmatrix} .42 & .02 \\ .08 & .48 \end{bmatrix} \begin{bmatrix} O \\ R \end{bmatrix}$$

Same e-vectors. E-values are .4, .5.

$$\begin{bmatrix} O \\ R \end{bmatrix} = 3 e^{.4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4 e^{.5t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$