Math 260 Fall 2022 **Exam 3** **November 21, 2022**

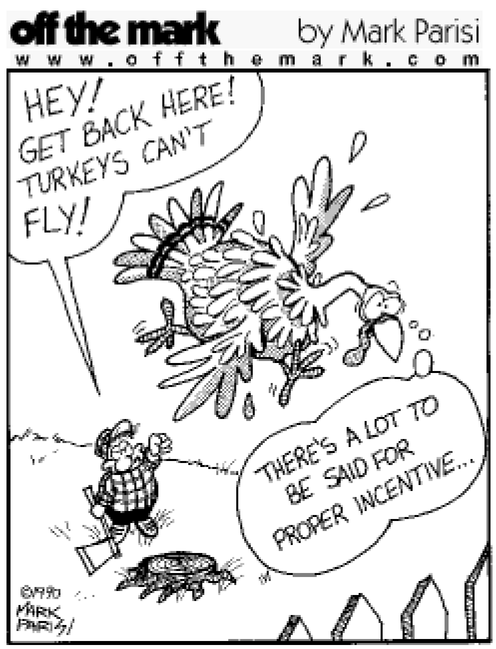
Name: ____

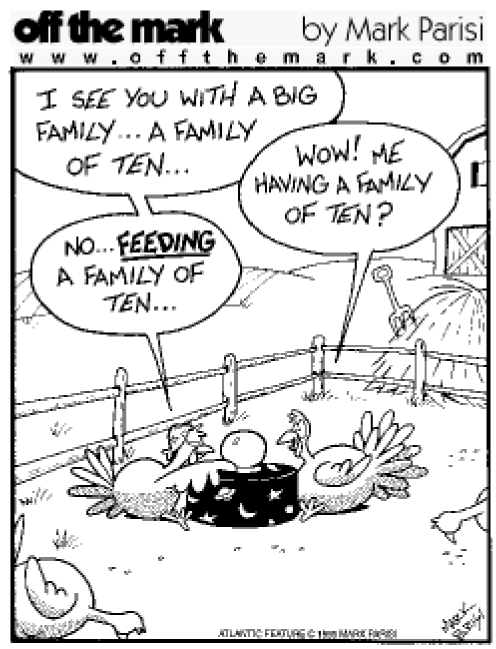
Problem	T/F	1	2	3 / 4	Total
Possible	40	32	12	16	100
Received					

DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 × 5 card of notes,

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.





40 points T/F. Answer the following 20 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may be true sometimes or under some circumstances, but not always or not necessarily. No explanation is necessary whether true or false.

True False If $W = span\{\vec{v}_1, \vec{v}_2\}$, then $Proj_W\vec{u} = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$.

True False $\vec{u} - Proj_W\vec{u}$ is parallel to W.

True False $A\vec{x} = \vec{b}$ always has a unique least squares solution.

True False If $\hat{\vec{x}}$ is the least squares solution to $A\vec{x} = \vec{b}$, then for any other $\vec{x}$ we have $\|\vec{b} - A\hat{\vec{x}}\| \leq \|\vec{b} - A\vec{x}\|$.

True False The angle between vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ -5 \\ 6 \end{bmatrix}$ is $90°$.

True False The distance between vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} + \vec{v}\|$.

True False For a square matrix A, vectors in $Col\ A$ are orthogonal to vectors in $Nul\ A^T$.

True False If $\|\vec{u} + \vec{v}\| < \|\vec{u}\| + \|\vec{v}\|$, then $\vec{u} \cdot \vec{v} \neq 0$.

True False A linearly independent set of non-zero vectors must be an orthogonal set.

True False An orthogonal set of non-zero vectors is linearly independent.

True False $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\right\}$ is an orthogonal basis for R^4.

True False If L is a line through the origin and if $\hat{\vec{y}}$ is the orthogonal projection of $\vec{y}$ onto L, then $\|\vec{y} - \hat{\vec{y}}\|$ gives the distance from $\vec{y}$ to L.

True False If the columns of A are orthonormal, then for any vector $\vec{x}$ we have $\|A\vec{x}\| = \|\vec{x}\|$.

True False If $Proj_{\vec{v}}\,\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}$, then $Proj_{3\vec{v}}\,\vec{u} = \begin{bmatrix}3\\6\end{bmatrix}$.

True False If $Proj_{\vec{v}}\,\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}$, then $Proj_{\vec{v}}\,3\vec{u} = \begin{bmatrix}3\\6\end{bmatrix}$.

True False If U and V are orthogonal matrices (if $U^TU = I$ and $V^TV = I$), then their product UV is orthogonal.

True False Given a linearly independent set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, dividing each by its own length $\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \dots, \frac{\vec{v}_n}{\|\vec{v}_n\|}\}$ will result in an orthonormal set of vectors.

True False If λ_1 and λ_2 are eigenvalues of $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, then $|\lambda_1| = 1$ and $|\lambda_2| = 1$.

True False If the columns of U form an orthonormal basis for R^n, then for $\vec{y} \in R^n$, $UU^T\vec{y} = \vec{y}$.

True False The projection of $\begin{bmatrix}1\\2\end{bmatrix}$ onto $span\{\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\}$ is $\begin{bmatrix}7/2\\1/2\end{bmatrix}$.

32 points 1. We are interested in finding the solution to

$$\begin{aligned} x + 3y &= 0 \\ x + 2y &= -1 \\ x + y &= 4 \end{aligned}$$

/7 With more equations than unknowns, likely this system does not have an exact solution. So let's do the best we can. First (and being careful with your arithmetic) find the least squares solution $\hat{\vec{x}}$ to $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}.$$

/2 Compute $A\hat{\vec{x}}$ using the above $\hat{\vec{x}}$ you just found. (Recall that $\hat{\vec{b}} = A\hat{\vec{x}} = Proj_{\,Col\,A}\,\vec{b}$ is the closest we can get to building vector $\vec{b}$ using the columns of A.)

/2 Find $\vec{b} - \hat{\vec{b}}$, and confirm that $\vec{b} - \hat{\vec{b}} \perp$ the columns of A.

Problem 1 continued (careful in doing your arithmetic!)

/6 We still want to find $Proj_{\,Col\,A}\,\vec{b}$, where $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$. But this time we'll do it a bit differently. First, use the Gram-Schmidt Process to construct two orthogonal vectors whose span is the same as the two columns of matrix A. That is, where $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, find $\vec{v}_1$ and $\vec{v}_2$ so that $span\{\vec{v}_1, \vec{v}_2\} = span\{\vec{u}_1, \vec{u}_2\}$, but where $\vec{v}_1 \perp \vec{v}_2$.

/7 Of course your $\vec{v}_1$ and $\vec{v}_2$ should be orthogonal. Using this fact, find $Proj_{\,\vec{v}_1, \vec{v}_2}\,\vec{b}$ (or if you prefer the alternate notation, find $Proj_{\,W}\,\vec{b}$, where $W = span\{\vec{v}_1, \vec{v}_2\}$).

/8 Using your work above, find the QR-factorization of A where Q has orthogonal columns and R is upper triangular. ($+1$ point extra credit if you find Q and R so that the columns of Q are ortho<u>normal</u>).

12 points 2. We'll determine what a 2×2 matrix A does to a vector by examining the long term behavior of multiplying some vector by $\boldsymbol{A}$ and by $\boldsymbol{A^{-1}}$.

Given initial vector $\vec{x}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, where $\vec{x}_{k+1} = \boldsymbol{A}\,\vec{x}_k$, we have (approximately)

k	0	1	⋯	10	11
$\vec{x}_k$	$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 7 \\ 4 \end{bmatrix}$	⋯	$\begin{bmatrix} 3 \times 10^5 \\ 3 \times 10^5 \end{bmatrix}$	$\begin{bmatrix} 9 \times 10^5 \\ 9 \times 10^5 \end{bmatrix}$

and (now using $\vec{x}_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$) where $\vec{x}_{k+1} = \boldsymbol{A^{-1}}\,\vec{x}_k$ we have (approximately)

k	0	1	⋯	26	27
$\vec{x}_k$	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.17 \\ 1.33 \end{bmatrix}$	⋯	$\begin{bmatrix} 2 \times 10^{-8} \\ 4 \times 10^{-8} \end{bmatrix}$	$\begin{bmatrix} 1 \times 10^{-8} \\ 2 \times 10^{-8} \end{bmatrix}$

Find $A^3\vec{x}$ where $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Hint: estimate the eigenvalues and eigenvectors from the above information, find $\vec{x}$ as a linear combination of those two eigenvectors, and then compute $A^3\vec{x}$.

/8 3. Given functions $f(t) = t$ and $g(t) = t^2$, use the Gram-Schmidt Process to find two functions that are orthogonal under the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$.

/8 4. Suppose that for two populations (let's say Owls O and Rats R), in month k their populations are O_k and R_k where

$$O_{k+1} = 1.42\, O_k + 0.02\, R_k$$
$$R_{k+1} = 0.08\, O_k + 1.48\, R_k$$

with initial populations of $\begin{bmatrix} O_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$. The eigenvectors of $\begin{bmatrix} 1.42 & 0.02 \\ 0.08 & 1.48 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ with corresponding eigenvalues of 1.4 and 1.5 respectively.

Find a general expression/formula for $\vec{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$.

/2 Extra credit. The above is the discrete version of this problem. What would the continuous version of this problem be and what would the corresponding solution be?