Math 260 Fall 2022 **Exam 2** **October 27, 2022**

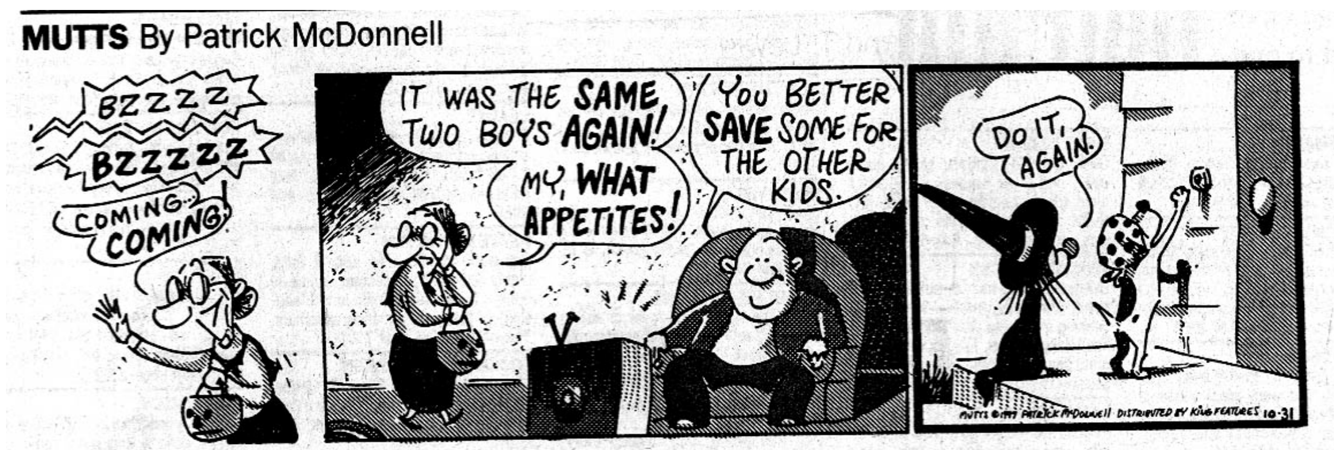
Name: Solutions

Problem	T/F	1	2 / 3	4	5	Total
Possible	40	26	14	12	8	100
Received						

DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 × 5 card of notes. You will not use a calculator.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.



40 points Answer the following 20 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may be true sometimes or under some circumstances, but not always or not necessarily. <u>No explanation is necessary</u> whether true or false.

True False The set of vectors of the form $\left\{\begin{bmatrix} a+1 \\ b \\ 2b \end{bmatrix}\right\}$ (for some a and b) is a subspace of R^3.

True False For a 2×2 matrix, if $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 4, then $\begin{bmatrix} -15 \\ -5 \end{bmatrix}$ is also an eigenvector with eigenvalue -20. still 4.

True False Let A be a 5×5 matrix with just two different eigenvalues. It is possible for A to have a complete set of 5 linearly independent eigenvectors.

True False If a 8×5 matrix has rank 2, then $nullity\ A = 3$.

True False For a 3×7 matrix A, since there are 4 more columns than rows, the largest possible dimension of $Nul\ A$ is 4. $4 \le \dim Nul\ A \le 7$

True False If $\det A \neq 0$, then $\det(A^{-1}) = \frac{1}{\det A}$.

True False For 3×3 matrix A, if $\det(A - \lambda I) = \lambda^3 - \lambda$, then A has no inverse.
$\lambda^3 - \lambda = 0 \Rightarrow$ one e-value is 0, so A is "bad."

True False If 3×3 matrix A has eigenvalues of 4, 5 and 6, then the eigenvectors of A form a basis for R^3.

True False For $v_1, v_2, v_3 \in R^4$, $Span\{v_1, v_2, v_3\}$ is a subspace of R^4.
Span{...} is always a subspace

True False It is possible that $\{v_1, v_3\}$ is a basis for $Span\{v_1, v_2, v_3\}$.
Example: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

True False If $A\vec{v}_1 = \lambda_1\vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$ where $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, then it must be that $\lambda_1 \neq \lambda_2$.

True False The sum of two eigenvectors of a matrix is an eigenvector of that same matrix.

Only if they have the same e-value.

True False There is a value of k for which $\begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix}$ has an eigenvalue of 0.

True False For 3×3 matrix A, $\det(-2A^T) = -6 \det A$.

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True False If the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 3$, then $\begin{vmatrix} a+2c & b+2d \\ c & d \end{vmatrix} = 3$.

True False The column space of A is the set of all vectors that can be written $A\vec{x}$ for some $\vec{x}$.

$\text{Col } A = \{\vec{b} : \vec{b} = A\vec{x} \text{ for some } \vec{x}\}$

True False If $\{\vec{v}_1, \ldots, \vec{v}_7\}$ spans R^7, then $\{\vec{v}_1, \ldots, \vec{v}_7\}$ is linearly independent.

True False If A is a 3×7 matrix, then $\dim Row\, A < \dim Col\, A$.

Always $=$

True False The set of matrices $\{\begin{bmatrix} a & a-b \\ b & c \end{bmatrix}\}$ (for some a, b and c) is a subspace of $M_{2\times2}$ (the set of all 2×2 matrices).

True False If 3×3 matrix A has rank 2, then $A\vec{x} = \vec{b}$ has an infinite number of solutions for some right hand side $\vec{b}$, no solution for some other right hand side $\vec{b}$, and a unique solution for yet another right hand side $\vec{b}$.

This last part is the only false part.

26 points 1. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Find where $A\vec{x} = \lambda \vec{x}$, etc.

/10 Find the eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of A.

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & 0-\lambda \end{vmatrix} = \cdots = (\lambda - 2)(\lambda + 1) = 0.$$

Now find $\vec{x}$ so that $A\vec{x} = \lambda\vec{x}$, i.e. $(A - \lambda I)\vec{x} = \vec{0}$.

$$\lambda_1 = 2 : \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right] \Rightarrow x_1 = 2x_2 \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 : \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right] \Rightarrow x_1 = -x_2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

/1 Once you found the eigenvalues λ_1 and λ_2 (and before you actually found the eigenvectors $\vec{v}_1$ and $\vec{v}_2$), how did you know that $\vec{v}_1$ and $\vec{v}_2$ would be linearly independent?

$\lambda_1 \neq \lambda_2$

/2 Give two different diagonalizations $PDP^{-1} = [\vec{v}_1 \ \vec{v}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [\vec{v}_1 \ \vec{v}_2]^{-1}$ of A.

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \text{ and } \begin{bmatrix} \pi & 2 \\ -\pi & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \pi & 2 \\ -\pi & 1 \end{bmatrix}^{-1}$$

/5 Let B be the basis for R^2 formed from eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Where $\vec{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, find $[\vec{x}]_B$, the co-ordinates of $\vec{x}$ with respect to B. (That is, find c_1 and c_2 so that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$.)

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \cdots = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

So $$\begin{bmatrix} 1 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

/8 Where $\vec{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, find $A^5\vec{x}$.

$$A^5\vec{x} = 2^5 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + (-1)^5 \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

$$= \cdots = \begin{bmatrix} 131 \\ 61 \end{bmatrix}.$$

5 points 2. Use Cramer's Rule to find x_3 in the linear system

$$\begin{aligned} 2x_1 \quad\quad\quad\quad &= 2 \\ 3x_1 + 5x_2 \quad\quad &= -2 \\ 4x_1 \quad\quad + x_3 &= 2 \end{aligned}.$$

$$\frac{\begin{vmatrix} 2 & 0 & 2 \\ 3 & 5 & -2 \\ 4 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 0 & 1 \end{vmatrix}} = \frac{5\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix}}{2 \cdot 5 \cdot 1} = \frac{-20}{10} = -2$$

10 points 3. Consider probability matrix $P = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$ and initial vector $\vec{x}_0 = \begin{bmatrix} 70 \\ 0 \end{bmatrix}$. Let $\vec{x}_k = A^k \vec{x}_0$.

/1 Find $\vec{x}_1 = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}\begin{bmatrix} 70 \\ 0 \end{bmatrix} = \begin{bmatrix} 42 \\ 28 \end{bmatrix}$

/5 Find the eigenvector of P that corresponds to eigenvalue 1. $P\vec{x} = \vec{x} \Rightarrow (P - I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} -.4 & .3 & 0 \\ .4 & -.3 & 0 \end{array}\right] \Rightarrow .4x_1 = .3x_2 \Rightarrow x_1 = \tfrac{3}{4}x_2$$

So e-vector is $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$, or any multiple, eg. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, or $\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$.

/2 Find P^∞. $= \begin{bmatrix} 3/7 & 3/7 \\ 4/7 & 4/7 \end{bmatrix}$

/2 Where $\underbrace{\vec{x}_0 = \begin{bmatrix} 70 \\ 0 \end{bmatrix}}_{\text{Sum is } 70}$, find $\vec{x}_\infty$. $\leftarrow$ multiple of $\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ with sum 70,

So $70 \cdot \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$.

12 points 4. Consider matrix A, which is row equivalent to matrix B:

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the following:

/2 $rank\ A =$ 3

/2 $\dim Nul\ A =$ 3

/2 A basis for $Row\ A$:

First three rows of either A or B.

/2 A basis for $Col\ A$:

Cols 1, 2 and 4 of A (not B)

/4 A basis for $Nul\ A$: Using B :

$$\begin{aligned} x_1 \qquad - 2x_3 \qquad + 9x_5 + 2x_6 &= 0 \\ x_2 - x_3 \qquad + 7x_5 + 3x_6 &= 0 \\ x_4 - x_5 - 2x_6 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

4 points 5. For $n \times n$ matrix A with eigenvalue λ, show that the eigenspace

$$\lambda(A) = \{ \vec{x} : A\vec{x} = \lambda\vec{x} \}$$

is a subspace of R^n.

Suppose $\vec{u}, \vec{v} \in \lambda(A)$. Then $A\vec{u} = \lambda\vec{u}$, $A\vec{v} = \lambda\vec{v}$.

Thus $A(c_1\vec{u} + c_2\vec{v}) = c_1 A\vec{u} + c_2 A\vec{v}$

$= c_1 \lambda\vec{u} + c_2 \lambda\vec{v}$

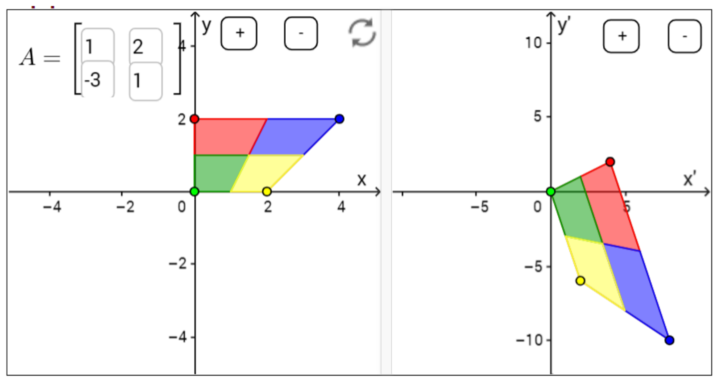
$= \lambda(c_1\vec{u} + c_2\vec{v})$

$\Rightarrow c_1\vec{u} + c_2\vec{v} \in \lambda(A)$. Done.

4 points 6. If the shape on the left is transformed into the shape on right using the transformation $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix},$$

what is the area of the shape on right?



Area of shape on right =

= Area on left * det A

= 6 · 7 = 42.