Math 260 Fall 2022	Exam 1	September 29, 2022

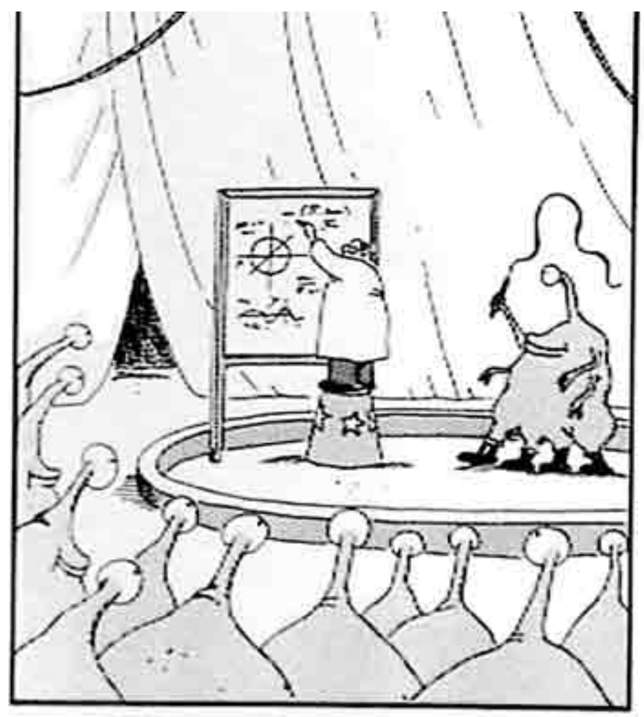
Name: ____

Problem	T/F	1 / 2 / 3	4 / 5	6	7	8	Total
Possible	40	17	13	10	9	11	100
Received							

You may use your one 3 x 5 card (both sides) of handwritten notes.
You may not use a calculator or any other materials or resources.

Be sure to show all pertinent work for each problem (not for True/False questions).

But boss, I just left out a decimal point. Don't I get at least partial credit?



Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.

40 points T/F. Answer the following 20 True/False questions. Each question is worth 2 points. Note: "True" means *always true* or *necessarily true*. "False" means that it may sometimes be true, but not always or not necessarily. *No work or explanation or justification is needed to be shown for these questions.*

True False $\begin{bmatrix} 1 & b \\ a & a^2 \end{bmatrix}$ has no inverse if $a = b$, and has an inverse otherwise.

$1 \cdot a^2 - a \cdot b = a(a-b) = 0 \Rightarrow a = b$ or $a = 0$.

True False The columns of a 2×3 (so 2 rows, 3 columns) could be linearly independent.

True False The columns of a 2×3 (so 2 rows, 3 columns) could span R^2.

Pretty likely.

True False The columns of a 2×3 (so 2 rows, 3 columns) must span R^2. Example: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

True False If A is a 2×3 matrix and B is a 3×4 matrix, then $(AB)^T(AB)$ is a 4×4 matrix.

True False If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, then $A^T = A$.

True False If $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, then $A^{-1} = A$.

True False If $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, then $A\vec{x} = \vec{b}$ must have more than one solution for some $\vec{b} \neq \vec{0}$.

Example: $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

True False Any vector that can be written as $A\vec{x}$ for some vector $\vec{x}$ is actually a linear combination of the columns of A.

True False If the columns of $n \times n$ A span R^n, then the columns are linearly independent.

Good A $\Rightarrow$ also good

True False If $Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

$\vec{v}_4$ must be a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

True False In R^3, if $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then $Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a plane in R^3.

True False $\begin{bmatrix} 7 \\ -3 \end{bmatrix}$ can be written as a linear combination of $\{\begin{bmatrix} 2.4 \\ -3.1 \end{bmatrix}, \begin{bmatrix} 5.6 \\ 6.5 \end{bmatrix}\}$.

They are lin. ind. and thus span $\mathbb{R}^2$.

True False For 3×3 matrix A, the second column of A^{-1} is the same as the solution $\vec{x}$ to the problem $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$$A \underbrace{[\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3]}_{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

True False If the columns of $n \times n$ matrix A are linearly independent, then $A\vec{x} = \vec{b}$ will have a solution, no matter what $\vec{b}$ is.

$\vec{x} = A^{-1}\vec{b}$

True False If the columns of $n \times n$ matrix A are linearly dependent, then $A\vec{x} = \vec{b}$ will not have a solution, no matter what $\vec{b}$ is.

True False For $A\vec{x} = \vec{b}$, it is possible that $A\vec{x} = \vec{b}_1$ has exactly one solution for some $\vec{b}_1$ while $A\vec{x} = \vec{b}_2$ has no solution for some other $\vec{b}_2$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$, $b_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. $\vec{b}_2 = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$

True False If $T(x_1, x_2) = (2x_1 + 3x_2, 4x_1 + 5x_2)$, then $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

True False If $n \times n$ matrices A, B, C and D are invertible, then $(ABCA)^{-1} = A^{-1}B^{-1}C^{-1}A^{-1}$.

True False The problem of solving for x_1 and x_2 in the system of equations

$$2x_1 + 5x_2 = 4$$
$$3x_1 + 6x_2 = 7$$

is equivalent of solving for x_1 and x_2 in the vector equation

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

6 points 1. Find the inverse of $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ by row reduction.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

6 points 2. Suppose for $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

you have $E_3 E_2 E_1 A = I$. Find A.

$$A^{-1} = E_3 E_2 E_1 \Rightarrow A = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \cdots = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

5 points 3. Find the inverse of $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$. Hint: think partitioned/block matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

So inverse is

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

8 points 4. I'm thinking of three numbers:

Their sum is 2.

The first number minus the second number is 3.

Set up a system of two equations and three unknowns which corresponds to this information. Find the numbers. If there is more than one possible set of numbers, find the general solution and give two specific solutions. If there is no solution, explain why.

$$x + y + z = 2$$
$$x - y = 3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \begin{aligned} x &= \tfrac{5}{2} - \tfrac{1}{2}z \\ y &= -\tfrac{1}{2} - \tfrac{1}{2}z \\ z &= z \end{aligned} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$z = 0: \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \qquad z = 1: \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \text{etc.}$$

5 points 5. Same conditions as Problem 4 above, but now with a third condition that

Twice the first number added to the third number is 6.

Find the numbers. If there is more than one possible set of numbers, find the general solution and give two specific solutions. If there is no solution, explain why.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 3 \\ 2 & 0 & 1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & 1 \\ 0 & -2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solution.

10 points 6. A company produces two items, but consumes some of each product in the production process. Given the consumption matrix

$$C = \begin{bmatrix} .2 & .4 \\ .6 & .2 \end{bmatrix}$$

how much should they produce if they want to end up with 30 units of each product?

$$(I - C)^{-1} = \begin{bmatrix} .8 & -.4 \\ -.6 & .8 \end{bmatrix}^{-1} = \cdots = \begin{bmatrix} 2 & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$$

$$* \begin{bmatrix} 30 \\ 30 \end{bmatrix} = \begin{bmatrix} 90 \\ 105 \end{bmatrix}$$

How much more of each product needs to be produced if demand for Product <u>Two</u> were to increase by 3 units (and demand for Product One does not change)?

$$* \begin{bmatrix} 30 \\ 33 \end{bmatrix} = \begin{bmatrix} 93 \\ 111 \end{bmatrix}, \text{ So } \begin{bmatrix} 3 \\ 6 \end{bmatrix} \text{ more}$$

OR $$* \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

OR $3 *$ Column 2 of $(I - C)^{-1}$.

9 points 7. Consider $A = \begin{bmatrix} 1 & -2 & 1 & -3 & -2 \\ 1 & -2 & 2 & -6 & -3 \\ -2 & 4 & -2 & 6 & 5 \end{bmatrix}$ which is the coefficient matrix for the system of 3 equations and 5 unknowns $A\vec{x} = \vec{b}$ with augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -3 & -2 & -3 \\ 1 & -2 & 2 & -6 & -3 & -4 \\ -2 & 4 & -2 & 6 & 5 & 12 \end{array}\right].$$

I've done all of the work to find that

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -3 & -2 & -3 \\ 1 & -2 & 2 & -6 & -3 & -4 \\ -2 & 4 & -2 & 6 & 5 & 12 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right].$$

/7 Give the general solution to $A\vec{x} = \vec{b}$ and give two specific solutions. Which part of your solution is the solution to the homogeneous problem $A\vec{x} = \vec{0}$?

$$\left.\begin{aligned} x_1 &= 4 + 2x_2 \\ x_2 &= x_2 \\ x_3 &= 5 + 3x_4 \\ x_4 &= x_4 \\ x_5 &= 6 \end{aligned}\right\} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

Homogeneous solutions

$x_2 = 0$, $x_4 = 0$: $\begin{pmatrix} 4 \\ 0 \\ 5 \\ 0 \\ 6 \end{pmatrix}$ $x_2 = 1$, $x_4 = 0$: $\begin{pmatrix} 6 \\ 1 \\ 5 \\ 0 \\ 6 \end{pmatrix}$ etc.

/1 What is it about the reduced row echelon matrix $\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$ that tells you that there is a solution to $A\vec{x} = \vec{b}$ no matter what the right hand side $\vec{b}$ is?

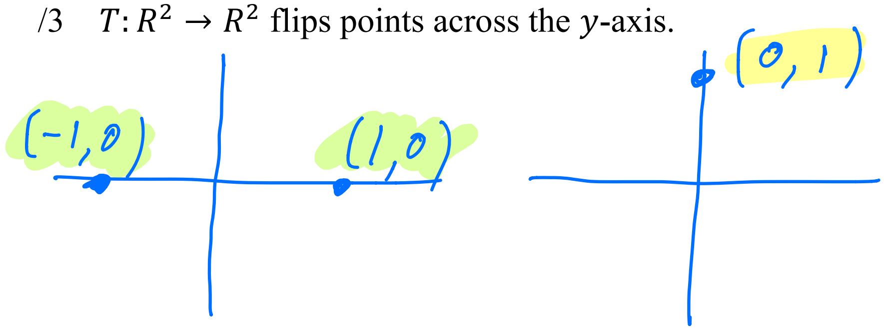
Pivot in every row

/1 Which columns of $\begin{bmatrix} 1 & -2 & 1 & -3 & -2 \\ 1 & -2 & 2 & -6 & -3 \\ -2 & 4 & -2 & 6 & 5 \end{bmatrix}$ form a linearly independent set?

Just circle the columns in this matrix.

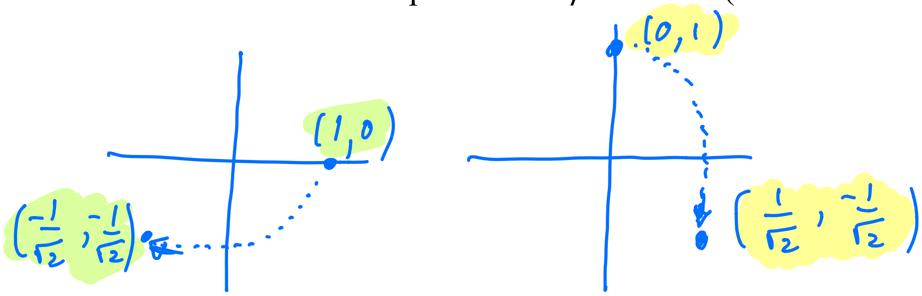
11 points 8. Assume that T is a linear transformation. Find the standard matrix of T.

/3 $T: R^2 \to R^2$ flips points across the y-axis.



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

/4 $T: R^2 \to R^2$ rotates points $-3\pi/4$ radians (so clockwise ↻).



$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

/4 $T: R^2 \to R^2$ flips points across the y-axis, and then rotates them $-3\pi/4$ radians.

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$