Section 5.6 The Normal Distribution

Math 141

<u>Main ideas</u>

We can measure/describe how extreme a value is by finding how many standard deviations away from the mean that value is. This is call a z-score or **z-value**:

$$z = \frac{x-\mu}{\sigma} \quad \text{which also means that } x = \mu + z \cdot \sigma.$$

The z-value of x is how many standard deviations σ above or below mean μ the value x is.

Normal (typical, usual) **distribution** (the outcomes and what fraction of the time each occurs):

- Many outcomes close to the mean outcome.
- Not so many outcomes away from the mean.
- Symmetry, balance

For normally distributed data, the probability that a particular value is less than or equal to a certain value x is $A(z)$, where z is the z-value of the value of x and $A(z)$ is the area "under the curve" below (to the left of) z, found in Table 2 on page 283. Excel and various other technology can also be used to find values.

<u>Problems</u>

1. Continuing from the Section 5.5 Handout:

Scores	μ	σ	Score of 85	Number of standard deviations score 85 is above the mean:
75, 78, 79, 80, 81, 82, 85	80	2.93	Great: top score!	
65, 70, 75, 80, 85, 90, 95	80	10	Good: above average	

2. Find the z-values for the seven exam scores in each case:

75, 78, 79, 80, 81, 82, 85

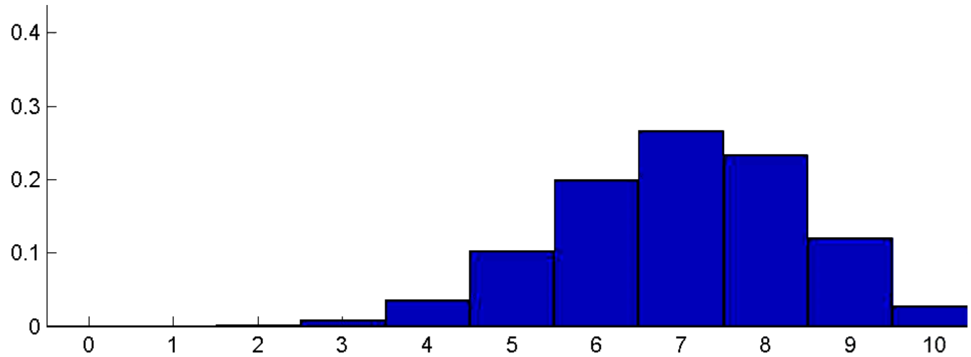
Score x	z-value
75	$z = \frac{75-80}{2.93} \approx -1.71$
78	
79	
80	
81	
82	
85	

65, 70, 75, 80, 85, 90, 95

Score x	z-value
65	$z = \frac{65-80}{10} = -1.5$
70	
75	
80	
85	
90	
95	

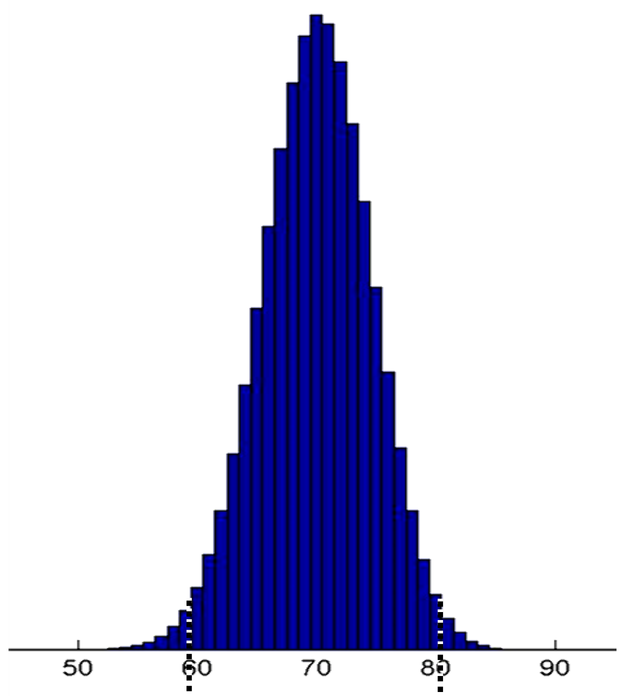
3. 70% free throw shooter. Shoot 10 shots.
   The probability distribution can be described with a table of values (X is the number of shots made) or with a histogram.

k	$Pr(X = k)$
0	.000006
1	.0001
2	.0014
3	.0090
4	.0368
5	.1029
6	.2001
7	.2668
8	.2335
9	.1211
10	.0282

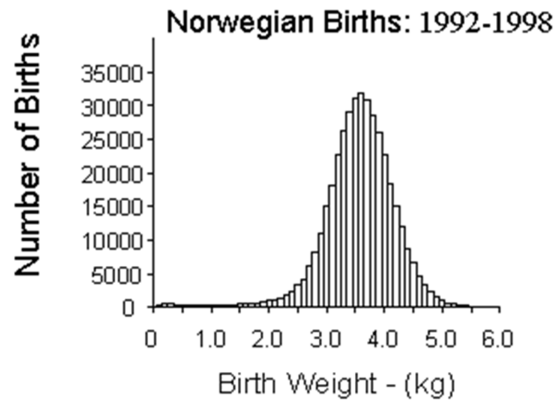


   Notice that area = probability.

4. Area = probability.
   70% free throw shooter.
   Shoot 100 shots.
   The probability of making between 60% and 80% of your shots is the area in that range.
   The total area "under the curve" is 1.

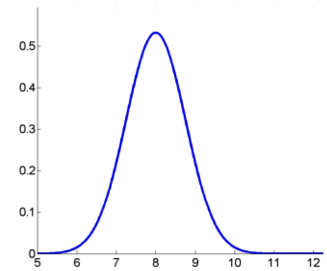


5. Births in Norway. Descriptive statistics:
   $\mu \approx 8$ pounds (about 3.6 kilograms)
   $\sigma \approx 0.75$ pounds (about .35 kilograms)
   Find the z-value for a weight of 9.5 pounds:



So a weight of 9.5 pounds is ______ standard deviations above the mean:

Let's modify the plot a bit: convert kilograms to pounds, change number of births to fraction of all births, and "smooth out" the histogram.



Let X = birth weight. The *fraction* of births within a certain range is the *area* under the curve in that range. For example, the fraction of births less than 9.5 pounds (i.e., the probability that a particular birth is less than 9.5 pounds) is the area under the curve from the 9.5 pounds and below. But how to compute that value exactly?

If the data are normally distributed, we can use technology of various sorts or Table 2 on page 283, or see more complete table at back of this handout.

6. For the same Norwegian birth weights problem, find the probabilities that for a given baby the birth weight would be:

   ≤ 9.5 pounds: $\frac{9.5-8.0}{.75} = \frac{1.5}{.75} = 2.0$ and $A(2.0) = .9772$.

   > 9.5 pounds:

   < 9.5 pounds:

   = 9.5 pounds:

   ≤ 8.75 pounds:

   ≤ 8.5 pounds:

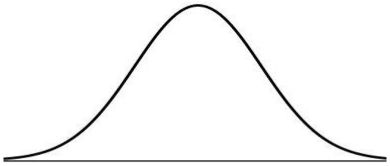
   or with Excel, `NORMSDIST(.5/.75) = 0.7475`

   ≤ 8 pounds:

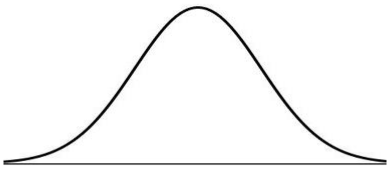
   ≤ 7.25 pounds:

   ≤ 6.5 pounds:

7. Suppose that the average height of male adults is 70 inches with standard deviation of 3 inches. What fraction of males are shorter than 76 inches?



Suppose that the average height of female adults is 64 inches with standard deviation of 2.5 inches. What fraction of females are shorter than 69 inches?



In general, what fraction of persons are shorter than the height that is 2 standard deviations above the mean height?

8. Coffee dispensing machine. Suppose on average $\mu = 11$ ounces and $\sigma = .4$ ounces. If the cups into which the coffee is being dispensed hold 12 ounces, what fraction of cups will be overfilled?

9. Suppose we have some normally distributed exam scores. Suppose your exam score is 85 and the average exam score is $\mu = 80$. What percentage of the exams is your 85 higher than if:

The standard deviation is $\sigma = 10$: $z = \frac{85-80}{10} =$ and $A(\quad) =$

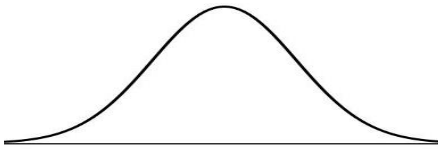
So your score of 85 is better than approximately _______% of the other scores.

The standard deviation is $\sigma = 2.93$:

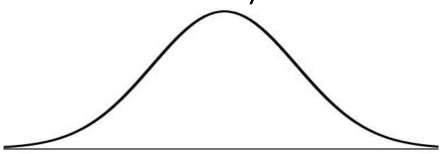
So your score of 85 is better than approximately _______% of the other scores.

10. Light bulbs. Average light bulb lasts $\mu = 1200$ hours with variation of $\sigma = 160$.

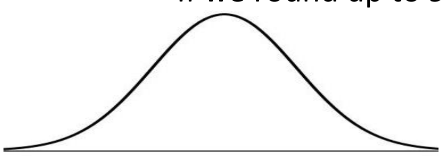
    What fraction of the bulbs burn out by the 1000 hour mark?



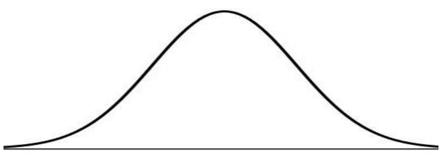
    What should the warranty be for how long the the bulbs last if we want to replace (at most) 5% of the bulbs? (In Excel, `NORMSINV(.05) = -1.645.`)



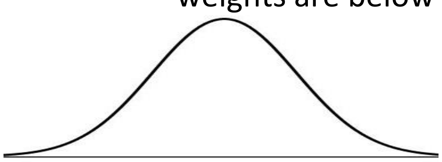
    If we round up to 950 hours, what fraction of bulbs will burn out by the 950 hour mark?



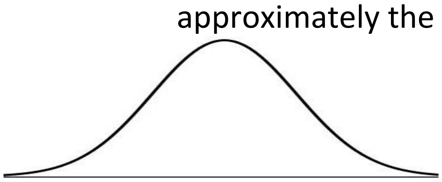
    If we round down to 900 hours, what fraction of bulbs will burn out by the 900 hour mark?

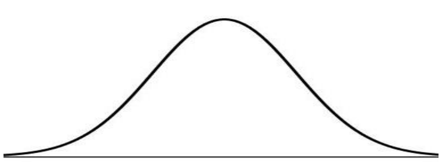


11. Norwegian birth weights. Average birth weight is $\mu = 8$ and $\sigma = .75$. What birth weight is at the 90^{th} percentile? That is, what is the weight for which 90% of all birth weights are below that weight?



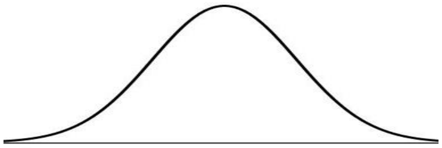
12. Percentiles for SAT scores. In 2013, the mean and standard deviation for SAT scores were approximately 1497 and 308. Assuming normal distribution (which is approximately the case), what SAT scores were at the 90^{th} and 99^{th} percentiles?





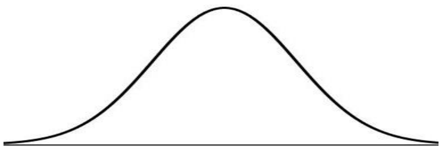
13. Fraction of values within a specified range.

    Light bulb problem: $\mu = 1200,\ \sigma = 160$. What fraction of light bulbs will burn out somewhere between 1000 hours and 1400 hours?

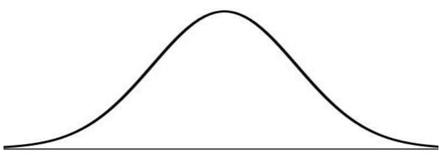


14. Fraction of values within 1 standard deviation (above or below) of the mean.

    Light bulb problem: $\mu = 1200,\ \sigma = 160$. What fraction of light bulbs will burn out somewhere between 1040 hours and 1360 hours?

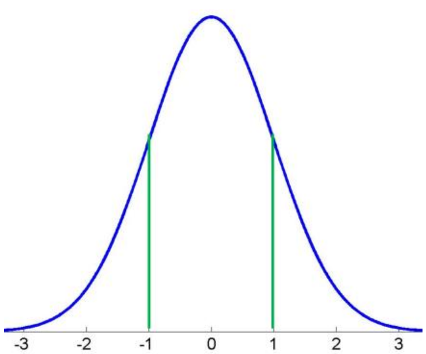


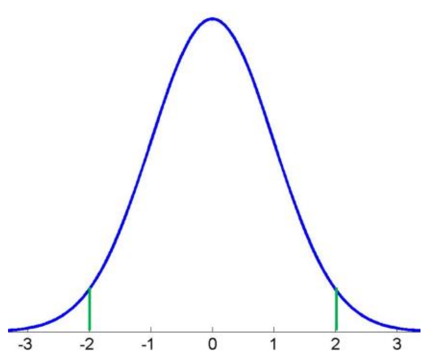
Coffee problem. $\mu = 11,\ \sigma = .4$. What fraction of cups will end up with between 10.6 ounces and 11.4 ounces of coffee?

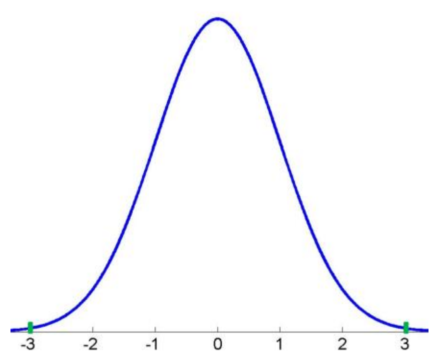


In general, what fraction of normally distributed data is within:

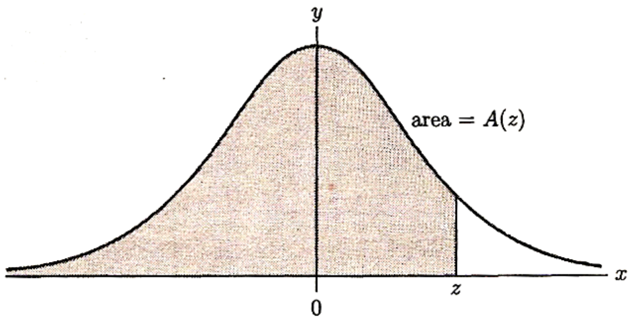
1 std. deviation of the mean? 2 std. deviations of the mean? 3 std. deviations of the mean?







A more complete table of z-values and corresponding areas.



Areas under the standard normal curve

z	$A(z)$	z	$A(z)$	z	$A(z)$	z	$A(z)$	z	$A(z)$
−3.50	.0002	−2.00	.0228	−.50	.3085	1.00	.8413	2.45	.9929
−3.45	.0003	−1.95	.0256	−.45	.3264	1.05	.8531	2.50	.9938
−3.40	.0003	−1.90	.0287	−.40	.3446	1.10	.8643	2.55	.9946
−3.35	.0004	−1.85	.0322	−.35	.3632	1.15	.8749	2.60	.9953
−3.30	.0005	−1.80	.0359	−.30	.3821	1.20	.8849	2.65	.9960
−3.25	.0006	−1.75	.0401	−.25	.4013	1.25	.8944	2.70	.9965
−3.20	.0007	−1.70	.0446	−.20	.4207	1.2813	.9000	2.75	.9970
−3.15	.0008	−1.65	.0495	−.15	.4404	1.30	.9032	2.80	.9974
−3.10	.0010	−1.60	.0548	−.10	.4602	1.35	.9115	2.85	.9978
−3.05	.0011	−1.55	.0606	−.05	.4801	1.40	.9192	2.90	.9981
−3.00	.0013	−1.50	.0668	.00	.5000	1.45	.9265	2.95	.9984
−2.95	.0016	−1.45	.0735	.05	.5199	1.50	.9332	3.00	.9987
−2.90	.0019	−1.40	.0808	.10	.5398	1.55	.9394	3.05	.9989
−2.85	.0022	−1.35	.0885	.15	.5596	1.60	.9452	3.10	.9990
−2.80	.0026	−1.30	.0968	.20	.5793	1.65	.9505	3.15	.9992
−2.75	.0030	−1.25	.1056	.25	.5987	1.70	.9554	3.20	.9993
−2.70	.0035	−1.20	.1151	.30	.6179	1.75	.9599	3.25	.9994
−2.65	.0040	−1.15	.1251	.35	.6368	1.80	.9641	3.30	.9995
−2.60	.0047	−1.10	.1357	.40	.6554	1.85	.9678	3.35	.9996
−2.55	.0054	−1.05	.1469	.45	.6736	1.90	.9713	3.40	.9997
−2.50	.0062	−1.00	.1587	.50	.6915	1.95	.9744	3.45	.9997
−2.45	.0071	−.95	.1711	.55	.7088	2.00	.9772	3.50	.9998
−2.40	.0082	−.90	.1841	.60	.7257	2.05	.9798		
−2.35	.0094	−.85	.1977	.65	.7422	2.10	.9821		
−2.30	.0107	−.80	.2119	.70	.7580	2.15	.9842		
−2.25	.0122	−.75	.2266	.75	.7734	2.20	.9861		
−2.20	.0139	−.70	.2420	.80	.7881	2.25	.9878		
−2.15	.0158	−.65	.2578	.85	.8023	2.30	.9893		
−2.10	.0179	−.60	.2743	.90	.8159	2.35	.9906		
−2.05	.0202	−.55	.2912	.95	.8289	2.40	.9918		

And of course we can use technology to find any values we want, not just the ones listed here.