Section 3.5/3.6 Permutations and Combinations/Further Counting Problems

Math 141

Main ideas

Permutation: order matters $P(n,r) = \frac{n!}{(n-r)!} = n \cdot (n-1) \cdot \cdots \cdot (n-r+1)$.

Example: $P(10,4) = \frac{10!}{(10-4)!} = \frac{10\cdot9\cdot8\cdot7\cdot6!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7$.

To *permute* means to *order* (put in a certain order).

Combination: order does not matter $C(n,r) = \frac{n!}{r!\,(n-r)!}$.

Example: $(10,4) = \frac{10!}{(10-4)!\,4!} = \frac{10!}{6!\,4!} = \frac{10\cdot9\cdot8\cdot7\cdot6!}{6!\cdot4\cdot3\cdot2\cdot1} = \frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1}$.

Two other common notations for $C(n,r)$ are $\binom{n}{r}$ and C_r^n.

Note that $P(n,r) = C(n,r) \cdot r!$ This means:

The number of ways to choose *and* order r *items from* n *total* $=$

The number of ways to choose r *items from* n $\cdot$ *The number of ways to* order *the* r *items.*

For both permutations and combinations, we say "n choose r." For example, "10 choose 4."

There are $r! = r \cdot (r-1) \cdot (r-2) \cdot \cdots \cdot 2 \cdot 1$ ways to order (put in a particular order) r items.

Examples: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, $1! = 1$ and $0! = 1$.

For computations you can use various technology, e.g. Excel, calculators or Google.

Problems

1. Factorials:

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ $\quad$ $\frac{9!}{7!} = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = \frac{9\cdot8\cdot7!}{7!} = 9 \cdot 8$ $\quad$ $C(9,2) = \frac{9!}{2!\,7!} = \frac{9\cdot8\cdot7!}{2\cdot1\cdot7!} = \frac{9\cdot8}{2\cdot1}$

2. Combinations:

$C(5,2) = \frac{5!}{2!\,3!} = \frac{5\cdot4}{2\cdot1}$ $\quad$ $C(5,3) = \frac{5!}{3!\,2!}$ (which $= C(5,2)$)

$C(5,0) = \frac{5!}{0!\,5!} = 1$ $\quad$ $C(5,1) = \frac{5!}{1!\,4!} = 5$

$C(n,0) = \frac{n!}{0!\,n!} = 1$ $\quad$ $C(n,1) = \frac{n!}{1!\,(n-1)!} = n$

$C(5,5) = \frac{5!}{0!\,5!} = 1$ $\quad$ $C(5,4) = \frac{5!}{4!\,1!} = 5$

$C(n,n) = \frac{n!}{n!\,0!} = 1$ $\quad$ $C(n,n-1) = \frac{n!}{(n-1)!\,1!} = n$

3. Three problems:

$P(20,3) = \frac{20!}{(20-3)!} = \frac{20\cdot 19\cdot 18\cdot 17!}{17!} = 20\cdot 19\cdot 18$

Number of ways to choose 3 persons from 20 if order matters: $P(20,3)$

Number of ways to award gold, silver & bronze medals if 20 athletes compete:

$\underline{20}\cdot\underline{19}\cdot\underline{18}$

4. Three other problems:

$C(20,3) = \frac{20!}{3!\,17!} = \frac{20\cdot 19\cdot 18\cdot 17!}{3\cdot 2\cdot 1\cdot 17!} = \frac{20\cdot 19\cdot 18}{3\cdot 2\cdot 1}$

Number of ways to choose 3 persons from 20 if order does not matter: $C(20,3)$

Number of ways to award three medals (all gold) if 20 athletes compete: $C(20,3)$

5. Number of ways are there to create a 9-person batting order from a 21-person baseball team:

$\underline{21}\cdot\underline{20}\cdot\underline{19}\cdot\underline{18}\cdot\underline{17}\cdot\underline{16}\cdot\underline{15}\cdot\underline{14}\cdot\underline{13} = P(21,9)$

6. Number of ways to choose the starting 5 players from a 13-person basketball team:

$C(13,5) = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9}{5\cdot 4\cdot 3\cdot 2\cdot 1}$

7. Number of ways to choose a president, vice-president and secretary from 10 persons if

We can choose anyone we want: $C(10,3)\cdot 3! = P(10,3) = \underline{10}\cdot\underline{9}\cdot\underline{8}$

James must be president: $\underline{1}\cdot\underline{9}\cdot\underline{8} = P(9,2) = C(9,2)\cdot 2!$

James must be one of the three chosen: $\underline{1}\cdot\underline{9}\cdot\underline{8} + \underline{9}\cdot\underline{1}\cdot\underline{8} + \underline{9}\cdot\underline{8}\cdot\underline{1}$

Or alternatively: $\underline{3}\cdot\underline{9}\cdot\underline{8}$

↑ What position for James

8. Number of ways can we choose a committee of 3 from 10 persons if

We can choose anyone we want: $C(10,3)$

James must be one of the committee members: $1\cdot C(9,2)$

9. A pizzeria offers 15 different toppings. Number of different pizzas possible:

$C(15,0) + C(15,1) + \cdots + C(15,15)$

OR: 2^{15}

10. Number of ways five Italian books and four novels be placed on a bookshelf if

The books can be placed in any order: $9!$

The Italian books must be kept together: $5\cdot 5!\cdot 4!$

OR: $5!\cdot 5!$

11. Of a family of 4, number of different combinations could come to dinner of

Size 0: $C(4,0) = 1$

Size 1: $C(4,1) = 4$

Size 2: $C(4,2) = 6$

Size 3: $C(4,3) = 4$

Size 4: $C(4,0) = 1$

Notice: $2^4 = 1 + 4 + 6 + 4 + 1$

12. From 4 sets total, number of subsets of size

0 of 4: 1

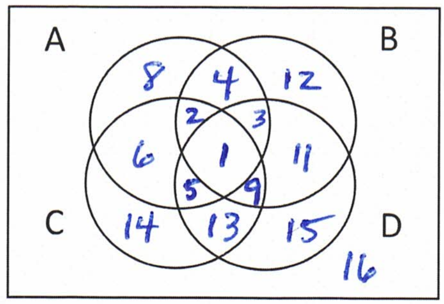
1 of 4: 4

2 of 4: 6

3 of 4: 4

4 of 4: 1

Notice: Same as in 11.



missing

	A	B	C	D
1	X	X	X	X
2	X	X	X	
3	X	X		X
4	X	X		
5	X		X	X
6	X		X	
7	X			X
8	X			
9		X	X	X
10		X	X	
11		X		X
12		X		
13			X	X
14			X	
15				X
16				

As seen in problems 11 and 12, in general: $C(n,0) + C(n,1) + \cdots + C(n,n) = 2^n$.

13. Suppose that 7 of 100 CDS are jazz. Number of ways to select 5 of the 100 in which

0 of the 5 is jazz: $C(7,0) \cdot C(93,5)$ = 51,971,283

1 of the 5 is jazz: $C(7,1) \cdot C(93,4)$ = 20,438,145

2 of the 5 are jazz: $C(7,2) \cdot C(93,3)$ = 2,725,086

3 of the 5 are jazz: $C(7,3) \cdot C(93,2)$ = 149,730

4 of the 5 are jazz: $C(7,4) \cdot C(93,1)$ = 3,255

All of the 5 are jazz: $C(7,5) \cdot C(93,0)$ = 21

Any number can be jazz or not: $C(100,5)$ = 75,287,520

For two scoop case, with flavors A,B,C,D,E:				
AA	BA	CA	DA	EA
AB	BB	CB	DB	EB
AC	BC	CC	DC	EC
AD	BD	CD	DD	ED
AE	BE	CE	DE	EE

14. An ice cream shop offers 5 flavors. Number of different cups of ice cream are possible of

1 scoop: 5

2 scoops: (Why isn't it simply $5 \cdot 5$?)

$\underbrace{5}_{\text{Same}} + \underbrace{C(5,2)}_{\text{Different}}$

3 scoops:

$\underbrace{5}_{\text{Same}} + \underbrace{C(5,2) \cdot 2}_{2/1} + \underbrace{C(5,3)}_{\text{Different}}$ or $P(5,2) = 5 \cdot 4$

15. Number of ways to choose 3 of 10 different toppings to put on

3 scoops of vanilla: $C(10,3)$

1 scoop of chocolate, 1 of vanilla, 1 of strawberry: $C(10,3) \cdot 3!$ or $\underbrace{10 \cdot 9 \cdot 8}_{P(10,3)}$

1 scoop of vanilla, 2 of chocolate:

$C(10,3) \cdot 3$ or $\dfrac{10 \cdot 9 \cdot 8}{2!}$

16. A restaurant offers its customers a choice of 3 side dishes with each meal. The side dishes can be chosen from a list of fifteen possibilities with duplications allowed. For instance, a customer can order two sides of mashed potatoes and one side of string beans. Show that there are 680 possible options for the three side dishes.

$\underbrace{15}_{\text{Same}} + \underbrace{C(15,2) \cdot 2}_{2/1} + \underbrace{C(15,3)}_{\text{Different}}$ or $P(15,2) = 15 \cdot 14$

An idea that comes up in Problem 17 below:

Recall that $P(n,r) = C(n,r) \cdot r!$

The number of ways to choose and order r items from n total $=$

The number of ways to choose r items from n $\cdot$ *The number of ways to order the r items.*

So $C(n,r) = \dfrac{P(n,r)}{r!}$

The number of ways to choose r items from n $=$

The number of ways to choose and order r items from n total / *The number of ways to order the r items.*

17. There are 2 senators from each of the 50 states. Number of ways to select a committee of 5 members if

The 5 could come from any state: $C(100,5) = \dfrac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

The 5 must come from 5 different states:

$\dfrac{100 \cdot 98 \cdot 96 \cdot 94 \cdot 92}{5!}$ or $C(50,5) \cdot 2^5 = \dfrac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{5!} \cdot 2^5$