Math 141 Fall 2024 **Midterm Exam 5** **November 22, 2024**

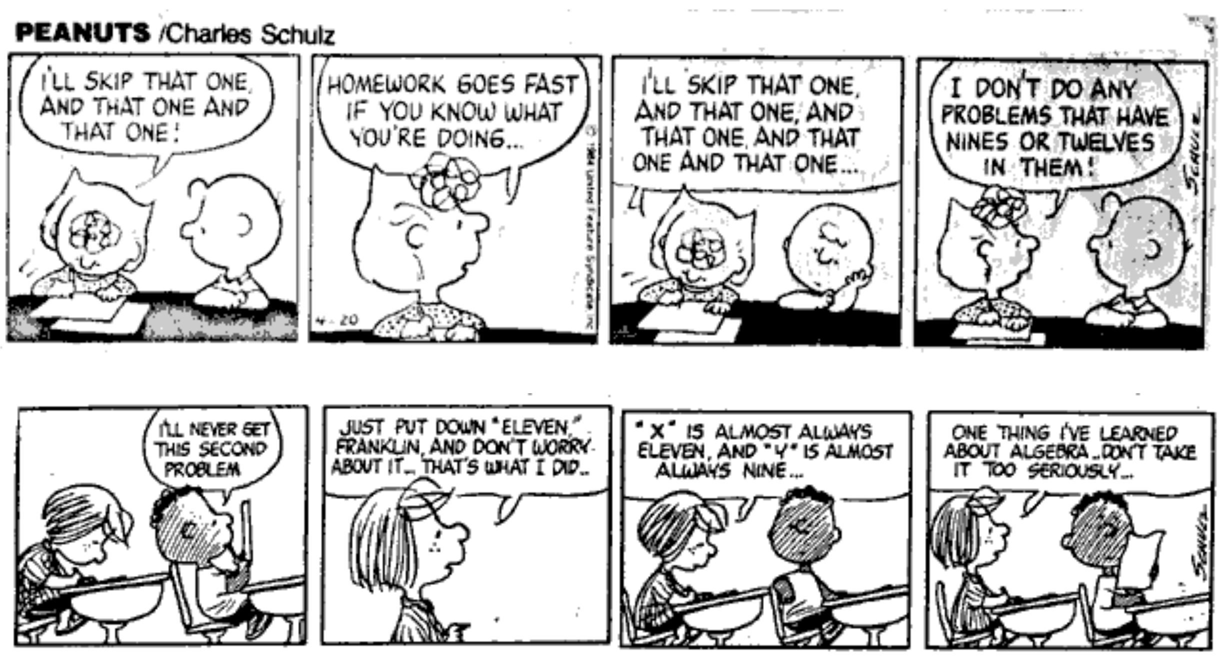
Name: Solutions

Problem	1	2	3/4	5	6	7	Total
Possible	20	14	20	16	15	15	100
Received							

DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 × 5 card of handwritten notes and a calculator.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.



20 points 1. Answer each of the following True/False questions. No explanation is needed.

T F The system of equations $\begin{aligned} x + 2y + 4z &= a \\ x + 3y + 5z &= 2 \end{aligned}$ will have an infinite number of solutions, no matter what a is. $\left[\begin{array}{ccc|c}1&2&4&a\\1&3&5&2\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&2&4&a\\0&1&1&2-a\end{array}\right]$

$\longrightarrow \left[\begin{array}{ccc|c}1&0&2&-a-4\\0&1&1&2-a\end{array}\right]$

T F The matrix $\begin{bmatrix} a & 1 \\ 6 & 2 \end{bmatrix}$ has an inverse no matter what a is.

Only if $2 \cdot a - 6 \cdot 1 \neq 0$

T F If matrix A does not have an inverse, then $AX = B$ will not have a solution.

T F For $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, the system $AX = B$ may not have a solution, depending on what B is.

It will have the solution $X = A^{-1}B$

T F A system of 5 equations and 5 unknowns could possibly have <u>no</u> solution.

T F A system of 5 equations and 5 unknowns could possibly have a <u>unique</u> solution.

T F A system of 3 equations and 2 unknowns could possibly have <u>infinite</u> solutions.

$x + y = 1$
$x + y = 1$
$x + y = 1$

T F A system of 3 equations and 2 unknowns could possibly have <u>no</u> solution.

T F A system of 2 equations and 3 unknowns could possibly have a <u>unique</u> solution.

If it has a solution, there will be a free variable, and thus infinite solutions.

T F A system of 2 equations and 3 unknowns could possibly have <u>infinite</u> solutions.

14 points 2. I'm thinking of three numbers:

The sum of the first and third numbers is 1.

The sum of the first and second numbers is 5.

The sum of the second and third numbers is 2.

Find the three numbers by coming up with the three equations that correspond to these three conditions, and then solve for the three numbers by performing Gauss-Jordan elimination on the resulting augmented matrix. (Don't just guess!)

$$\begin{aligned} x \quad\quad + z &= 1 \\ x + y \quad\quad &= 5 \\ y + z &= 2 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 5 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R2 - R1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R3 - R2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

$\frac{1}{2}R3$, then

$$\xrightarrow[R2 + R3]{R1 - R3} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \end{array} \begin{array}{c} x \\ y \\ z \end{array}$$

10 points 3. For what value(s) of $\boldsymbol{k}$ will
$$\begin{aligned} 2x + 4y &= 14 \\ 3x + 8y &= 25 \\ x + \boldsymbol{k}y &= 1 \end{aligned}$$
have at least one solution?

$$\left[\begin{array}{cc|c} 2 & 4 & 14 \\ 3 & 8 & 25 \\ 1 & k & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R1} \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 3 & 8 & 25 \\ 1 & k & 1 \end{array}\right] \xrightarrow[R3 - R1]{R2+(-3)R1} \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 2 & 4 \\ 0 & k-2 & -6 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R2} \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & k-2 & -6 \end{array}\right] \xrightarrow[R3+(2-k)R2]{R1+(-2)R2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -2k-2 \end{array}\right]$$

Need: $-2k-2 = 0 \Rightarrow k = -1.$

What is the solution (or what are the solutions, if there are more than one)?

$x = 3$

$y = 2$

10 points 4. A 100-seat movie theater charges \$20 for adults and \$10 for kids. If the theater is full and \$1700 is collected, how many adults and how many kids are in the theater? Set up two equations with two unknowns and either use Gauss-Jordan elimination or else the inverse of the corresponding 2×2 matrix to find your solution.

$$\begin{aligned} a + k &= 100 \\ 20a + 10k &= 1700 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 100 \\ 20 & 10 & 1700 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -10 & -300 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 70 \\ 0 & 1 & 30 \end{array}\right]$$

$a = 70$

$k = 30$

16 points 5. Suppose a certain economy consists of two sectors. Suppose that this economy has the input-output (consumption) matrix $A = \begin{bmatrix} .1 & .3 \\ .4 & .2 \end{bmatrix}$.

/2 How much of each product would be *consumed* if you *produced* 10 units of each product?

$$\begin{bmatrix} .1 & .3 \\ .4 & .2 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

/2 How much of each product is *remaining* if you *produced* 10 units of each product?

$$\begin{bmatrix} 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

/2 How much *more* of each product would be *consumed* if you produced *one more unit* of product 2?

Column 2 of A.

$$\begin{bmatrix} .1 & .3 \\ .4 & .2 \end{bmatrix}\begin{bmatrix} 10 \\ 11 \end{bmatrix} = \begin{bmatrix} 4.3 \\ 6.2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} + \begin{bmatrix} .3 \\ .2 \end{bmatrix}$$

/8 How much would you need to produce in order to *end up* with 10 units of each product? (Use the formula for finding the 2 × 2 matrix in this problem.)

$$(I-A)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .1 & .3 \\ .4 & .2 \end{bmatrix}\right)^{-1} = \begin{pmatrix} .9 & -.3 \\ -.4 & .8 \end{pmatrix}^{-1}$$

$$= \frac{1}{\underbrace{(.9)(.8)-(-.4)(-.3)}_{.60}}\begin{bmatrix} .8 & .3 \\ .4 & .9 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{3}{2} \end{bmatrix}$$

So need to produce $\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 18\frac{1}{3} \\ 21\frac{2}{3} \end{bmatrix}$

/2 How much *more* of each product would be needed if you wanted to *end up* with *ten more* units of product 2 (so 10 units of product 1, and 20 units of product 2).

$$\begin{bmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} = 10\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

15 points 6. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

/12 Use Gauss-Jordan elimination to find the inverse of A.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3-R2]{R1+(-2)R2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 2 & 0 \\ 0 & 1 & 3 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R2+(3)R3]{R1+(-3)R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -3 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

A^{-1}

/3 Use A^{-1} to find the solution to
$$\begin{aligned} x + 2y + 3z &= 5 \\ x + y &= 5 \\ y + 2z &= 1 \end{aligned}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 2 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

15 points 7. Find the solutions to each of the following systems. If a system has more than one solution, give the general solution and then give at least two specific solutions. If a system has no solution, state that (and show/explain why).

/6

$$x + y - 2z + 3w = 5$$
$$-2x - 2y + 2z + 2w = 8$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 5 \\ -2 & -2 & 2 & 2 & 8 \end{array}\right] \xrightarrow{R2 + 2R1} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 5 \\ 0 & 0 & -2 & 8 & 18 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 5 \\ 0 & 0 & 1 & -4 & -9 \end{array}\right] \xrightarrow{R1 + 2R2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -5 & -13 \\ 0 & 0 & 1 & -4 & -9 \end{array}\right]$$

$$x + y - 5w = -13$$
$$z - 4w = -9$$

General

$x = -y - 5w - 13$

$y = \text{free}$

$z = 4w - 9$

$w = \text{free}$

Specific

-13	-9
0	1
-9	-5
0	1

etc.

/4

$$3x + 6y = 21$$
$$-5x - 10y = 21$$

$$\left[\begin{array}{cc|c} 3 & 6 & 21 \\ -5 & -10 & 21 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 7 \\ -5 & -10 & 21 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 0 & 56 \end{array}\right]$$

No solution.

/5

$$n + d = 12$$
$$n - 5d = 0$$
$$5n + 10d = 70$$

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 1 & -5 & 0 \\ 5 & 10 & 70 \end{array}\right] \xrightarrow[R3 + (-5)R1]{R2 - R1} \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & -6 & -12 \\ 0 & 5 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 1 & 2 \\ 0 & 1 & 10 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$n = 10$

$d = 2$.