Math 141 Fall 2024 **Midterm Exam 2** **November 5, 2024**

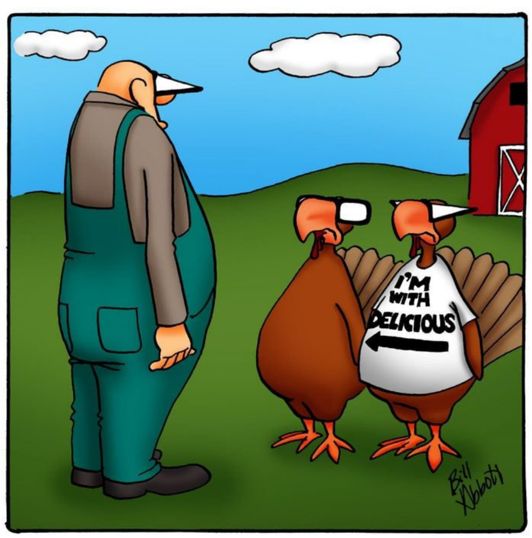
Name: Solutions

Problem	1	2	3 / 4 / 5	6	7	Total
Possible	13	24	30	16	17	100
Received						

DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

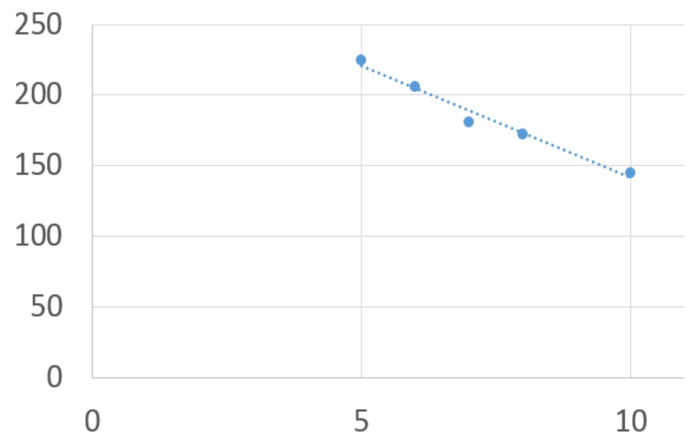
You may use a 3 × 5 card of handwritten notes and a calculator.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.



13 points 1. Suppose that a movie theater experimented in its pricing by setting the ticket price at five different levels and then recording how many tickets were sold at that price, as listed below.

Ticket price	Number of tickets sold
\$5	225
\$6	205
\$7	180
\$8	175
\$10	140



The line that best fits these data is approximately

$$y = -15x + 300$$

where x is ticket price and y is the number of tickets sold.

/4 What do the values of -15 and 300 mean?
(Don't just say slope and y-intercept—describe what each number tells us).

If $x \uparrow 1$, then $\downarrow 15$: raise ticket prices by \$1, sell 15 fewer tickets.

If $x = 0$, $y = 300$: If tickets were free, they could sell (i.e. give away) 300 tickets.

/3 According to this model, how many tickets would be sold at a price of \$9?

$y = -15(9) + 300 = 165$

/3 According to this model, at what price would no one buy any tickets?

$0 = -15x + 300$

$\Rightarrow x = \$20$

/3 According to this model, what should the price be to sell 240 tickets?

$240 = -15x + 300$

$\Rightarrow x = \$4.$

24 points 2. Consider the production function $f(x,y) = 100\sqrt{x^2+4y^2} = 100(x^2+4y^2)^{1/2}$, which gives the number of units produced as a function when using x units of labor and y units of capital.

/8 Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial x}(x,y) = 100\left(\tfrac{1}{2}\right)(x^2+4y^2)^{-\frac{1}{2}}\,2x = \frac{100x}{\sqrt{x^2+4y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = 100\left(\tfrac{1}{2}\right)(x^2+4y^2)^{-\frac{1}{2}}\,8y = \frac{400y}{\sqrt{x^2+4y^2}}$$

/12 Evaluate and (very briefly) interpret each of the following:

$$f(3,2) = 100\sqrt{3^2+4\cdot 2^2} = 100\sqrt{25} = 500$$

Interpretation: Production level at $(x,y) = (\overset{\text{labor}}{3}, \overset{\text{capital}}{2})$

$$\frac{\partial f}{\partial x}(3,2) = \frac{100\cdot 3}{\sqrt{25}} = 60$$

Interpretation: If $x \uparrow 1$, then $f \uparrow 60$ (approximately)

$$\frac{\partial f}{\partial y}(3,2) = \frac{400\cdot 2}{\sqrt{25}} = 160$$

Interpretation: If $y \uparrow 1$, then $f \uparrow 160$ (approx.)

/4 With $(3,2)$ of labor and capital, what is the marginal productivity of <u>capital</u>; that is, by approximately how much would production increase if <u>capital</u> were to *increase* by 1 unit?

With $(3,2)$ of labor and capital, by approximately how much would production increase if capital were to *decrease* by 3 units?

If $y \downarrow 1$, then $f \downarrow 160$ (approx.)

so if $y \downarrow 3$, then $f \downarrow \underbrace{3\cdot 160}_{540}$ (approx.)

6 points 3. In a certain city, most people take either the train or the bus to get to work. Let $f(p_B, p_T)$ be the number of people who take the bus, as a function of the price p_B of riding the bus and of the price p_T of riding the train.

/3 Should $\frac{\partial f}{\partial p_B}$ be > 0 or < 0 or $= 0$? Why? If $p_B \uparrow$, then $f \downarrow$

/3 Should $\frac{\partial f}{\partial p_T}$ be > 0 or < 0 or $= 0$? Why? If $p_T \uparrow$, then $f \uparrow$

12 points 4. Find the following derivatives.

/6 $\frac{\partial}{\partial y}\ x^3y^4 + 5x^{1/3} + \underbrace{\frac{5}{y^2}}^{5y^{-2}} + e^{xyz}$ $= x^3 \cdot 4y^3 + 0 + 5(-2)y^{-3} + e^{xyz} \cdot xz \leftarrow \frac{\partial}{\partial y} xyz$

/6 $\frac{\partial^2}{\partial x\,\partial y}\ x^3y^4 + 5x^{1/3} + \frac{5}{y^2} + e^{xyz}$

$= \frac{\partial}{\partial x}\left(4x^3y^3 - 10y^{-3} + e^{xyz} \cdot xz\right) = 12x^2y^3 - 0 + e^{xyz} \cdot yz \cdot xz + e^{xyz} \cdot z$ Product Rule

12 points 5. For the function

$$f(x, y) = x^3 + y^2 - 3x - 10y + 4$$

use first derivatives to find where $f(x, y)$ has critical points, and use the second derivative test to determine the type of each point.

$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$\frac{\partial f}{\partial y} = 2y - 10 = 0 \Rightarrow y = 5$

$(-1, 5), (1, 5)$

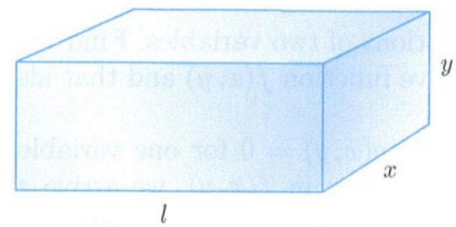
$D(x,y) = 6x \cdot 2 - 0 \cdot 0 = 12x$

$D(-1, 5) = 12(-1) < 0$, so neither min nor max

$D(1, 5) = 12(1) > 0$, so either min or max,

and $\frac{\partial^2 f}{\partial x^2}(1, 5) = 6(1) > 0$, so min.

16 points 6. Suppose that you want to mail a small package and that the U.S. Postal Service allows packages (that qualify for the "small package rate") with combined dimensions of 30 inches, that is, $x + y + l = 30$ (which means $l = 30 - x - y$). What are the dimensions of the box with maximum volume xyl that satisfy this restriction on combined dimensions?



Use first derivatives to determine the dimensions and second derivatives to verify that you have found the dimensions that do indeed maximize the volume. Show all work.

$f(x,y) = xy(30-x-y) = 30xy - x^2y - xy^2$

$\frac{\partial f}{\partial x} = 30y - 2xy - y^2 = y(30-2x-y) = 0 \Rightarrow y = 0$ (not possible) or $30 - 2x - y = 0$

$\frac{\partial f}{\partial y} = 30x - x^2 - 2xy = x(30-x-2y) = 0 \Rightarrow x = 0$ (not possible) or $30 - x - 2y = 0$

So $\begin{array}{l} 2x + y = 30 \\ x + 2y = 30 \end{array} \Rightarrow \cdots \Rightarrow \begin{array}{l} x = 10 \\ y = 10 \end{array}$ (and $l = 30 - 10 - 10 = 10$)

$D(x,y) = (-2y)(-2x) - (30 - 2x - 2y)^2$

$= (-2\cdot 10)(-2\cdot 10) - (30 - 2\cdot 10 - 2\cdot 10)^2$

$= 400 - 100 > 0$

so min or max.

$\frac{\partial^2 f}{\partial x^2} = -2\cdot 10 < 0 \Rightarrow$ max.

17 points 7. Suppose that x units of labor and y units of capital can produce $300x^{1/3}y^{2/3}$ units of a certain product and that the unit costs are \$10 for labor and \$20 for capital, so that the total cost is $10x + 20y$.

Suppose that we need a *production level* of 1,500. How many units of labor and how many units of capital should be used to *minimize cost* $10x + 20y$?

Use the Method of Lagrange Multipliers and show all pertinent work.

Constraint: $300x^{1/3}y^{2/3} = 1500$

i.e. $1500 - 300x^{1/3}y^{2/3} = 0$

$$F(x,y,\lambda) = 10x + 20y + \lambda\left[1500 - x^{1/3}y^{2/3}\right]$$

$$= 10x + 20y + 1500\lambda - \lambda x^{1/3}y^{2/3}$$

$$\frac{\partial F}{\partial x} = 10 - 300\lambda\left(\tfrac{1}{3}\right)x^{-2/3}y^{2/3} = 10 - 100\lambda\frac{y^{2/3}}{x^{2/3}}$$

$$\frac{\partial F}{\partial y} = 20 - 300\lambda x^{1/3}\left(\tfrac{2}{3}\right)y^{-1/3} = 20 - 200\lambda\frac{x^{1/3}}{y^{1/3}}$$

$$= 0 \Rightarrow 100\lambda\frac{y^{2/3}}{x^{2/3}} = 10 \Rightarrow \lambda = \frac{10x^{2/3}}{100y^{2/3}} = \frac{x^{2/3}}{10y^{2/3}}$$

$$= 0 \Rightarrow 200\lambda\frac{x^{1/3}}{y^{1/3}} = 20 \Rightarrow \lambda = \frac{20y^{1/3}}{200x^{1/3}} = \frac{y^{1/3}}{10x^{1/3}}$$

$$\text{So } \frac{x^{2/3}}{10y^{2/3}} = \frac{y^{1/3}}{10x^{1/3}} \Rightarrow 10\underbrace{x^{2/3}\cdot x^{1/3}}_{x} = 10\underbrace{y^{2/3}\cdot y^{1/3}}_{y}$$

so $y = x$

In constraint:

$$300x^{1/3}y^{2/3} = 1500$$

$$\Rightarrow \underbrace{x^{1/3}x^{2/3}}_{x} = 5$$

$$\Rightarrow y = 5.$$