Functions of Several Variables

Most of our applications of calculus have involved functions of one variable. In real life, however, a quantity of interest often depends on more than one variable. For instance, the sales level of a product may depend not only on its price, but also on the prices of competing products, the amount spent on advertising, and perhaps the time of year. The total cost of manufacturing the product depends on the cost of raw materials, labor, plant maintenance, and so on.

This chapter introduces the basic ideas of calculus for functions of more than one variable. Section 1 presents two examples that will be used throughout the chapter. Derivatives are treated in Section 2 and then used in Sections 3 and 4 to solve optimization problems more general than those you may have encountered before. The final two sections are devoted to least-squares problems and a brief introduction to the integration of functions of two variables.

1 Examples of Functions of Several Variables

A function $f(x, y)$ of the two variables x and y is a rule that assigns a number to each pair of values for the variables; for instance,

$$f(x, y) = e^x(x^2 + 2y).$$

An example of a function of three variables is

$$f(x, y, z) = 5xy^2z.$$

EXAMPLE 1 A Function with Two Variables A store sells butter at \$2.50 per pound and margarine at \$1.40 per pound. The revenue from the sale of x pounds of butter and y pounds of margarine is given by the function

$$f(x, y) = 2.50x + 1.40y.$$

Determine and interpret $f(200, 300)$.

SOLUTION $f(200, 300) = 2.50(200) + 1.40(300) = 500 + 420 = 920$. The revenue from the sale of 200 pounds of butter and 300 pounds of margarine is \$920. ▶ *Now Try Exercise 1*

From Chapter 7 of *Calculus & Its Applications*, Thirteenth Edition. Larry J. Goldstein, David C. Lay, David I. Schneider, Nakhlé H. Asmar.

A function $f(x, y)$ of two variables may be graphed in a manner analogous to that for functions of one variable. It is necessary to use a three-dimensional coordinate system, where each point is identified by three coordinates (x, y, z). For each choice of x, y, the graph of $f(x, y)$ includes the point $(x, y, f(x, y))$. This graph is usually a surface in three-dimensional space, with equation $z = f(x, y)$. (See Fig. 1.) Three graphs of specific functions are shown in Fig. 2.

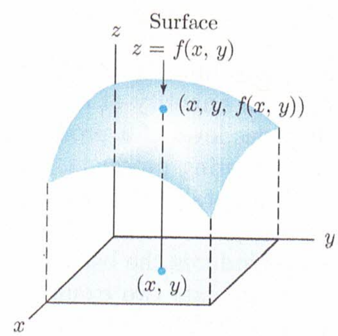


Figure 1 Graph of $f(x, y)$.

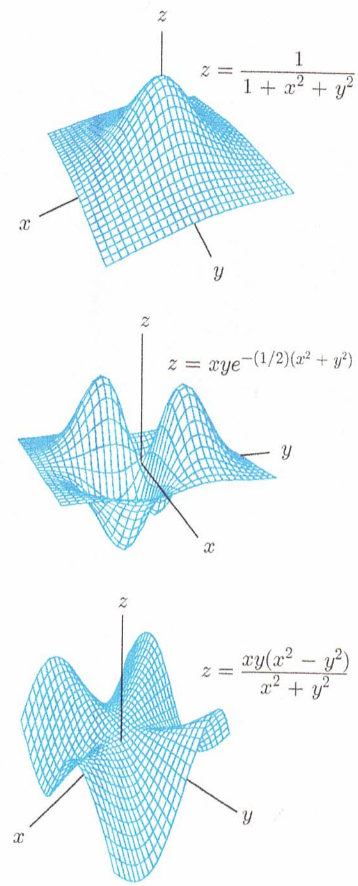


Figure 2

Application to Architectural Design

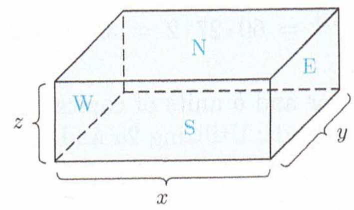
When designing a building, we would like to know, at least approximately, how much heat the building loses per day. The heat loss affects many aspects of the design, such as the size of the heating plant, the size and location of duct work, and so on. A building loses heat through its sides, roof, and floor. How much heat is lost will generally differ for each face of the building and will depend on such factors as insulation, materials used in construction, exposure (north, south, east, or west), and climate. It is possible to estimate how much heat is lost per square foot of each face. Using these data, we can construct a heat-loss function as in the following example.

EXAMPLE 2

Heat-Loss Function A rectangular industrial building of dimensions x, y, and z is shown in Fig. 3(a). In Fig. 3(b), we give the amount of heat lost per day by each side of the building, measured in suitable units of heat per square foot. Let $f(x, y, z)$ be the total daily heat loss for such a building.

(a) Find a formula for $f(x, y, z)$.

(b) Find the total daily heat loss if the building has length 100 feet, width 70 feet, and height 50 feet.



(a)

	Roof	East side	West side	North side	South side	Floor
Heat loss (per sq ft)	10	8	6	10	5	1
Area (sq ft)	xy	yz	yz	xz	xz	xy

(b)

Figure 3 Heat loss from an industrial building.

SOLUTION (a) The total heat loss is the sum of the amount of heat loss through each face of the building. The heat loss through the roof is

$$[\text{heat loss per square foot of roof}] \cdot [\text{area of roof in square feet}] = 10xy.$$

Similarly, the heat loss through the east side is $8yz$. Continuing in this way, we see that the total daily heat loss is

$$f(x, y, z) = 10xy + 8yz + 6yz + 10xz + 5xz + 1 \cdot xy.$$

We collect terms to obtain

$$f(x, y, z) = 11xy + 14yz + 15xz.$$

(b) The amount of heat loss when $x = 100$, $y = 70$, and $z = 50$ is given by $f(100, 70, 50)$, which equals

$$f(100, 70, 50) = 11(100)(70) + 14(70)(50) + 15(100)(50)$$
$$= 77{,}000 + 49{,}000 + 75{,}000 = 201{,}000.$$

▶ *Now Try Exercise 7*

In Section 3, we will determine the dimensions x, y, z that minimize the heat loss for a building of specific volume.

Production Functions in Economics The costs of a manufacturing process can generally be classified as one of two types: cost of labor and cost of capital. The meaning of the cost of labor is clear. By the cost of capital, we mean the cost of buildings, tools, machines, and similar items used in the production process. A manufacturer usually has some control over the relative portions of labor and capital utilized in its production process. It can completely automate production so that labor is at a minimum or utilize mostly labor and little capital. Suppose that x units of labor and y units of capital are used. (Economists normally use L and K, respectively, for labor and capital. However, for simplicity, we use x and y.) Let $f(x, y)$ denote the number of units of finished product that are manufactured. Economists have found that $f(x, y)$ is often a function of the form

$$f(x, y) = Cx^A y^{1-A},$$

where A and C are constants, $0 < A < 1$. Such a function is called a *Cobb–Douglas production function.*

EXAMPLE 3 **Production in a Firm** Suppose that, during a certain time period, the number of units of goods produced when x units of labor and y units of capital are used is $f(x, y) = 60x^{3/4}y^{1/4}$.

(a) How many units of goods will be produced by 81 units of labor and 16 units of capital?

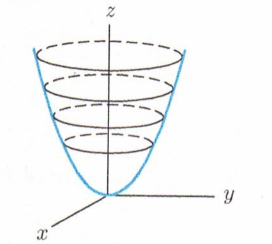
(b) Show that, whenever the amounts of labor and capital being used are doubled, so is the production. (Economists say that the production function has "constant returns to scale.")

SOLUTION (a) $f(81, 16) = 60(81)^{3/4} \cdot (16)^{1/4} = 60 \cdot 27 \cdot 2 = 3240$. There will be 3240 units of goods produced.

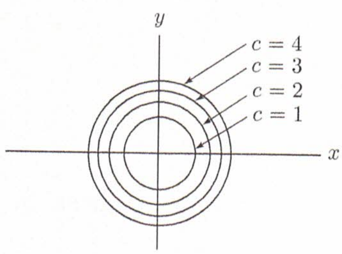
(b) Utilization of a units of labor and b units of capital results in the production of $f(a, b) = 60a^{3/4}b^{1/4}$ units of goods. Utilizing $2a$ and $2b$ units of labor and capital, respectively, results in $f(2a, 2b)$ units produced. Set $x = 2a$ and $y = 2b$. Then, we see that

$$
\begin{aligned}
f(2a, 2b) &= 60(2a)^{3/4}(2b)^{1/4} \\
&= 60 \cdot 2^{3/4} \cdot a^{3/4} \cdot 2^{1/4} \cdot b^{1/4} \\
&= 60 \cdot 2^{(3/4+1/4)} \cdot a^{3/4}b^{1/4} \\
&= 2^1 \cdot 60a^{3/4}b^{1/4} \\
&= 2f(a, b).
\end{aligned}
$$

▶ **Now Try Exercise 9**



Graph of $f(x, y) = x^2 + y^2$



Level curves of $f(x, y) = x^2 + y^2$

Figure 4 Level curves.

Level Curves It is possible graphically to depict a function $f(x, y)$ of two variables using a family of curves called *level curves*. Let c be any number. Then, the graph of the equation $f(x, y) = c$ is a curve in the xy-plane called the *level curve of height c*. This curve describes all points of height c on the graph of the function $f(x, y)$. As c varies, we have a family of level curves indicating the sets of points on which $f(x, y)$ assumes various values c. In Fig. 4, we have drawn the graph and various level curves for the function $f(x, y) = x^2 + y^2$.

Level curves often have interesting physical interpretations. For example, surveyors draw *topographic maps* that use level curves to represent points having equal altitude. Here $f(x, y) =$ the altitude at point (x, y). Figure 5(a) shows the graph of $f(x, y)$ for a typical hilly region. Figure 5(b) shows the level curves corresponding to various altitudes. Note that when the level curves are closer together the surface is steeper.

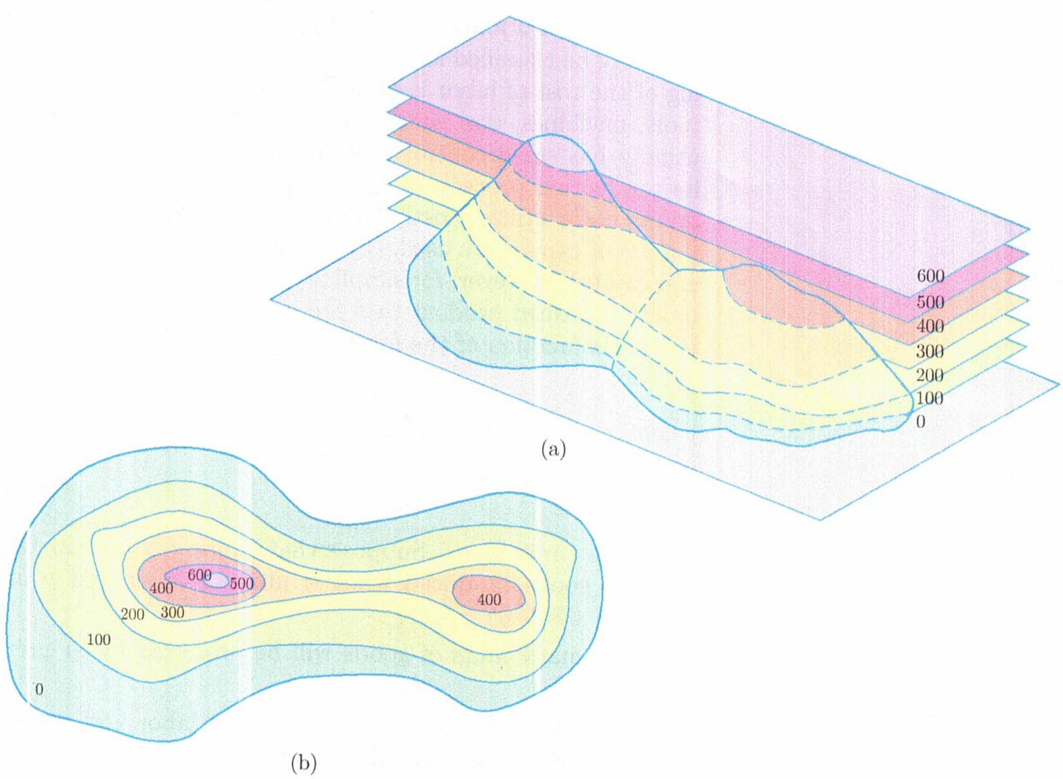


Figure 5 Topographic level curves show altitudes.

EXAMPLE 4 **Level Curves** Determine the level curve at height 600 for the production function $f(x, y) = 60x^{3/4}y^{1/4}$ of Example 3.

SOLUTION The level curve is the graph of $f(x, y) = 600$, or

$$60x^{3/4}y^{1/4} = 600$$

$$y^{1/4} = \frac{10}{x^{3/4}}$$

$$y = \frac{10{,}000}{x^3}.$$

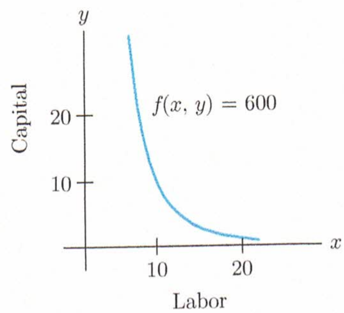


Figure 6 A level curve of a production function.

Of course, since x and y represent quantities of labor and capital, they must both be positive. We have sketched the graph of the level curve in Fig. 6. The points on the curve are precisely those combinations of capital and labor that yield 600 units of production. Economists call this curve an *isoquant*. ▶ **Now Try Exercise 15**

Check Your Understanding 1

1. Let $f(x, y, z) = x^2 + y/(x - z) - 4$. Compute $f(3, 5, 2)$.
2. In a certain country the daily demand for coffee is given by $f(p_1, p_2) = 16p_1/p_2$ thousand pounds, where p_1 and p_2 are the respective prices of tea and coffee in dollars per pound. Compute and interpret $f(3, 4)$.

EXERCISES 1

1. Let $f(x, y) = x^2 - 3xy - y^2$. Compute $f(5, 0)$, $f(5, -2)$, and $f(a, b)$.
2. Let $g(x, y) = \sqrt{x^2 + 2y^2}$. Compute $g(1, 1)$, $g(0, -1)$, and $g(a, b)$.
3. Let $g(x, y, z) = x/(y - z)$. Compute $g(2, 3, 4)$ and $g(7, 46, 44)$.
4. Let $f(x, y, z) = x^2 e^{\sqrt{y^2+z^2}}$. Compute $f(1, -1, 1)$ and $f(2, 3, -4)$.
5. Let $f(x, y) = xy$. Show that $f(2 + h, 3) - f(2, 3) = 3h$.
6. Let $f(x, y) = xy$. Show that $f(2, 3 + k) - f(2, 3) = 2k$.
7. **Cost** Find a formula $C(x, y, z)$ that gives the cost of materials for the closed rectangular box in Fig. 7(a), with dimensions in feet. Assume that the material for the top and bottom costs \$3 per square foot and the material for the sides costs \$5 per square foot.

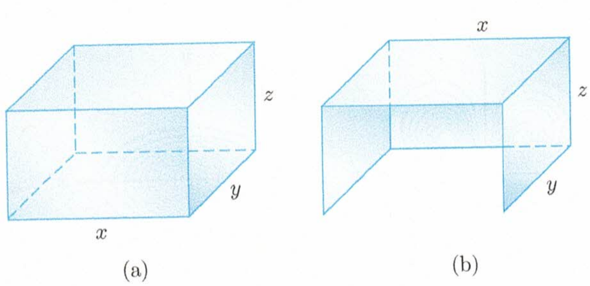


Figure 7

8. **Cost** Find a formula $C(x, y, z)$ that gives the cost of material for the rectangular enclosure in Fig. 7(b), with dimensions in feet. Assume that the material for the top costs \$3 per square foot and the material for the back and two sides costs \$5 per square foot.
9. Consider the Cobb–Douglas production function $f(x, y) = 20x^{1/3}y^{2/3}$. Compute $f(8, 1)$, $f(1, 27)$, and $f(8, 27)$. Show that, for any positive constant k, $f(8k, 27k) = kf(8, 27)$.
10. Let $f(x, y) = 10x^{2/5}y^{3/5}$. Show that $f(3a, 3b) = 3f(a, b)$.
11. **Present Value** The present value of A dollars to be paid t years in the future (assuming a 5% continuous interest rate) is $P(A, t) = Ae^{-.05t}$. Find and interpret $P(100, 13.8)$.
12. Refer to Example 3. If labor costs \$100 per unit and capital costs \$200 per unit, express as a function of two variables, $C(x, y)$, the cost of utilizing x units of labor and y units of capital.
13. **Tax and Homeowner Exemption** The value of residential property for tax purposes is usually much lower than its actual market value. If v is the market value, the *assessed value* for real estate taxes might be only 40% of v. Suppose that the property tax, T, in a community is given by the function

$$T = f(r, v, x) = \frac{r}{100}(.40v - x),$$

where v is the estimated market value of a property (in dollars), x is a *homeowner's exemption* (a number of dollars depending on the type of property), and r is the tax rate (stated in dollars per hundred dollars) of net assessed value.

(a) Determine the real estate tax on a property valued at \$200,000 with a homeowner's exemption of \$5000, assuming a tax rate of \$2.50 per hundred dollars of net assessed value.

(b) Determine the tax due if the tax rate increases by 20% to \$3.00 per hundred dollars of net assessed value. Assume the same property value and homeowner's exemption. Does the tax due also increase by 20%?

14. **Tax and Homeowner Exemption** Let $f(r, v, x)$ be the real estate tax function of Exercise 13.

(a) Determine the real estate tax on a property valued at \$100,000 with a homeowner's exemption of \$5000, assuming a tax rate of \$2.20 per hundred dollars of net assessed value.

(b) Determine the real estate tax when the market value rises 20% to \$120,000. Assume the same homeowner's exemption and a tax rate of \$2.20 per hundred dollars of net assessed value. Does the tax due also increase by 20%?

Draw the level curves of heights 0, 1, and 2 for the functions in Exercises 15 and 16.

15. $f(x, y) = 2x - y$ **16.** $f(x, y) = -x^2 + 2y$

17. Draw the level curve of the function $f(x, y) = x - y$ containing the point $(0, 0)$.

18. Draw the level curve of the function $f(x, y) = xy$ containing the point $(\frac{1}{2}, 4)$.

19. Find a function $f(x, y)$ that has the line $y = 3x - 4$ as a level curve.

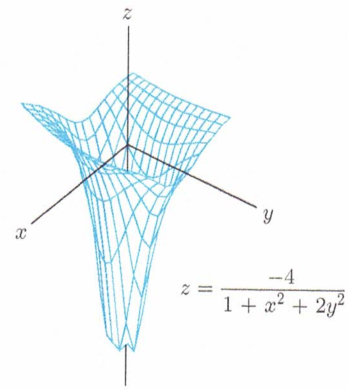
20. Find a function $f(x, y)$ that has the curve $y = 2/x^2$ as a level curve.

21. Suppose that a topographic map is viewed as the graph of a certain function $f(x, y)$. What are the level curves?

22. **Isocost Lines** A certain production process uses labor and capital. If the quantities of these commodities are x and y, respectively, the total cost is $100x + 200y$ dollars. Draw the level curves of height 600, 800, and 1000 for this function. Explain the significance of these curves. (Economists frequently refer to these lines as *budget lines* or *isocost lines*.)

Match the graphs of the functions in Exercises 23–26 to the systems of level curves shown in Figs. 8(a)–(d).

23.



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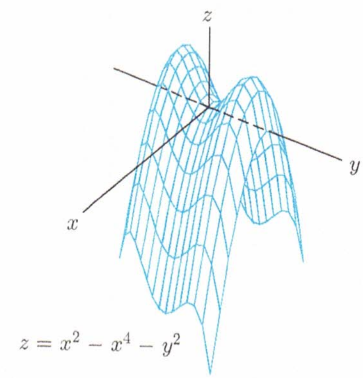

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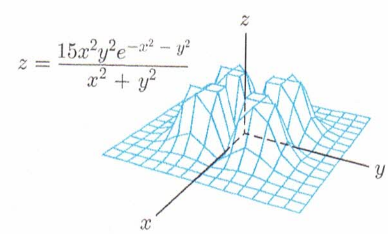

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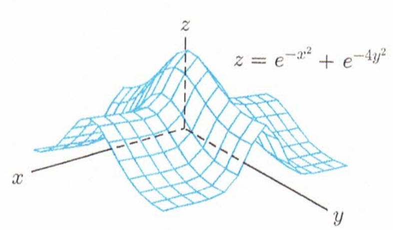


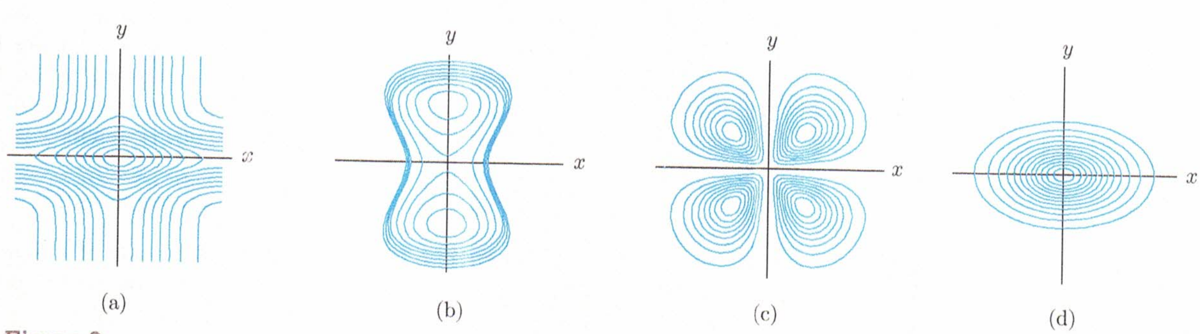


Figure 8

Solutions to Check Your Understanding 1

1. Substitute 3 for x, 5 for y, and 2 for z.

$$f(3,5,2) = 3^2 + \frac{5}{3-2} - 4 = 10$$

2. To compute $f(3,4)$, substitute 3 for p_1 and 4 for p_2 into $f(p_1, p_2) = 16p_1/p_2$. Thus,

$$f(3,4) = 16 \cdot \tfrac{3}{4} = 12.$$

Therefore, if the price of tea is \$3 per pound and the price of coffee is \$4 per pound, 12,000 pounds of coffee will be sold each day. (Notice that as the price of coffee increases the demand decreases.)

2 Partial Derivatives

We have introduced the notion of a derivative to measure the rate at which a function $f(x)$ is changing with respect to changes in the variable x. Let us now study the analog of the derivative for functions of two (or more) variables.

Let $f(x,y)$ be a function of the two variables x and y. Since we want to know how $f(x,y)$ changes with respect to the changes in both the variable x and the variable y, we shall define two derivatives of $f(x,y)$ (to be called partial derivatives), one with respect to each variable.

DEFINITION The *partial derivative of $f(x,y)$ with respect to x*, written $\frac{\partial f}{\partial x}$, is the derivative of $f(x,y)$, where y is treated as a constant and $f(x,y)$ is considered a function of x alone. The *partial derivative of $f(x,y)$ with respect to y*, written $\frac{\partial f}{\partial y}$, is the derivative of $f(x,y)$, where x is treated as a constant.

EXAMPLE 1 **Computing Partial Derivatives** Let $f(x,y) = 5x^3y^2$. Compute

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}.$$

SOLUTION To compute $\frac{\partial f}{\partial x}$, we think of $f(x,y)$ written as

$$f(x,y) = \left[5y^2\right] x^3,$$

where the brackets emphasize that $5y^2$ is to be treated as a constant. Therefore, when differentiating with respect to x, $f(x,y)$ is just a constant times x^3. If k is any constant, then

$$\frac{d}{dx}(kx^3) = 3 \cdot k \cdot x^2.$$

Thus,

$$\frac{\partial f}{\partial x} = 3 \cdot [5y^2] \cdot x^2 = 15x^2y^2.$$

After some practice, it is unnecessary to place the y^2 in front of the x^3 before differentiating.

Now, to compute $\frac{\partial f}{\partial y}$, we think of

$$f(x,y) = [5x^3]y^2.$$

When we are differentiating with respect to y, $f(x,y)$ is simply a constant (that is, $5x^3$) times y^2. Hence,

$$\frac{\partial f}{\partial y} = 2 \cdot [5x^3] \cdot y = 10x^3y.$$

▶ Now Try Exercise 1

EXAMPLE 2 **Computing Partial Derivatives** Let $f(x, y) = 3x^2 + 2xy + 5y$. Compute

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}.$$

SOLUTION To compute $\frac{\partial f}{\partial x}$, we think of

$$f(x, y) = 3x^2 + [2y]x + [5y].$$

Now, we differentiate $f(x, y)$ as if it were a quadratic polynomial in x:

$$\frac{\partial f}{\partial x} = 6x + [2y] + 0 = 6x + 2y.$$

Note that we treat $5y$ as a constant when differentiating with respect to x, so the partial derivative of $5y$ with respect to x is zero.

To compute $\frac{\partial f}{\partial y}$, we think of

$$f(x, y) = [3x^2] + [2x]y + 5y.$$

Then,

$$\frac{\partial f}{\partial y} = 0 + [2x] + 5 = 2x + 5.$$

Note that we treat $3x^2$ as a constant when differentiating with respect to y, so the partial derivative of $3x^2$ with respect to y is zero. ▶ *Now Try Exercise 3*

EXAMPLE 3 **Differentiation Rules and Partial Derivatives** Compute

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

for each of the following.

(a) $f(x, y) = (4x + 3y - 5)^8$ (b) $f(x, y) = e^{xy^2}$ (c) $f(x, y) = y/(x + 3y)$

SOLUTION (a) To compute $\frac{\partial f}{\partial x}$, we think of

$$f(x, y) = (4x + [3y - 5])^8.$$

By the general power rule,

$$\frac{\partial f}{\partial x} = 8 \cdot (4x + [3y - 5])^7 \cdot 4 = 32(4x + 3y - 5)^7.$$

Here, we used the fact that the derivative of $4x + 3y - 5$ with respect to x is just 4.

To compute $\frac{\partial f}{\partial y}$, we think of

$$f(x, y) = ([4x] + 3y - 5)^8.$$

Then,

$$\frac{\partial f}{\partial y} = 8 \cdot ([4x] + 3y - 5)^7 \cdot 3 = 24(4x + 3y - 5)^7.$$

(b) To compute $\frac{\partial f}{\partial x}$, we observe that

$$f(x, y) = e^{x[y^2]},$$

so that

$$\frac{\partial f}{\partial x} = [y^2]\, e^{x[y^2]} = y^2 e^{xy^2}.$$

To compute $\frac{\partial f}{\partial y}$, we think of

$$f(x, y) = e^{[x]y^2}.$$

Thus,

$$\frac{\partial f}{\partial y} = e^{[x]y^2} \cdot 2[x]y = 2xye^{xy^2}.$$

(c) To compute $\frac{\partial f}{\partial x}$, we use the general power rule to differentiate $[y](x+[3y])^{-1}$ with respect to x:

$$\frac{\partial f}{\partial x} = (-1)\cdot[y](x+[3y])^{-2}\cdot 1 = -\frac{y}{(x+3y)^2}.$$

To compute $\frac{\partial f}{\partial y}$, we use the quotient rule to differentiate

$$f(x,y) = \frac{y}{[x]+3y}$$

with respect to y. We find that

$$\frac{\partial f}{\partial y} = \frac{([x]+3y)\cdot 1 - y\cdot 3}{([x]+3y)^2} = \frac{x}{(x+3y)^2}.$$

The use of brackets to highlight constants is helpful initially when we compute partial derivatives. From now on, we shall merely form a mental picture of those terms to be treated as constants and dispense with brackets. ▶ *Now Try Exercise 9*

A partial derivative of a function of several variables is also a function of several variables and hence can be evaluated at specific values of the variables. We write

$$\frac{\partial f}{\partial x}(a,b)$$

for $\frac{\partial f}{\partial x}$ evaluated at $x=a$, $y=b$. Similarly,

$$\frac{\partial f}{\partial y}(a,b)$$

denotes the function $\frac{\partial f}{\partial y}$ evaluated at $x=a$, $y=b$.

EXAMPLE 4 **Evaluating Partial Derivatives** Let $f(x,y) = 3x^2 + 2xy + 5y$.

(a) Calculate $\frac{\partial f}{\partial x}(1,4)$. **(b)** Evaluate $\frac{\partial f}{\partial y}$ at $(x,y) = (1,4)$.

SOLUTION

(a) $\frac{\partial f}{\partial x} = 6x + 2y$, $\frac{\partial f}{\partial x}(1,4) = 6\cdot 1 + 2\cdot 4 = 14$

(b) $\frac{\partial f}{\partial y} = 2x + 5$, $\frac{\partial f}{\partial y}(1,4) = 2\cdot 1 + 5 = 7$ ▶ *Now Try Exercise 19*

Geometric Interpretation of Partial Derivatives Consider the three-dimensional surface $z = f(x,y)$ in Fig. 1. If y is held constant at b and x is allowed to vary, the equation

$$z = f(x, \underset{\text{constant}}{b})$$

describes a curve on the surface. [The curve is formed by cutting the surface $z = f(x,y)$ with a vertical plane parallel to the xz-plane.] The value of $\frac{\partial f}{\partial x}(a,b)$ is the slope of the tangent line to the curve at the point where $x=a$ and $y=b$.

Likewise, if x is held constant at a and y is allowed to vary, the equation

$$z = f(\underset{\text{constant}}{a}, y)$$

describes the curve on the surface $z = f(x,y)$ shown in Fig. 2. The value of the partial derivative $\frac{\partial f}{\partial y}(a,b)$ is the slope of this curve at the point where $x=a$ and $y=b$.

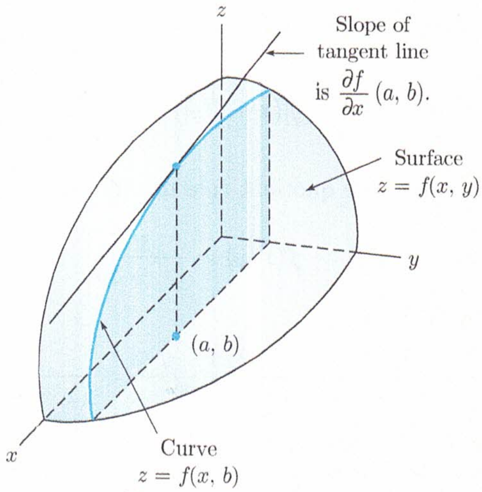


Figure 1 $\frac{\partial f}{\partial x}$ gives the slope of a curve formed by holding y constant.

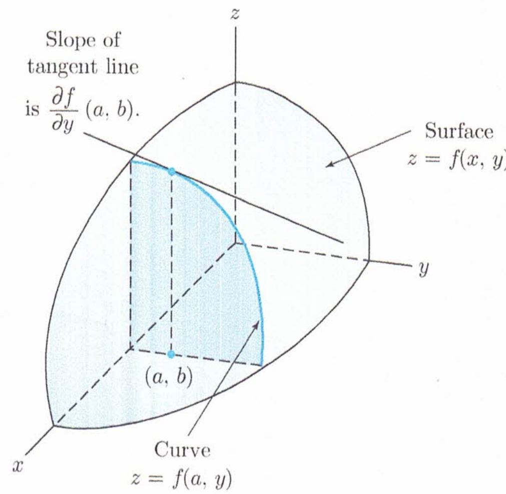


Figure 2 $\frac{\partial f}{\partial y}$ gives the slope of a curve formed by holding x constant.

Partial Derivatives and Rates of Change Since $\dfrac{\partial f}{\partial x}$ is simply the ordinary derivative with y held constant, $\dfrac{\partial f}{\partial x}$ gives the rate of change of $f(x, y)$ with respect to x for y held constant. In other words, keeping y constant and increasing x by one (small) unit produces a change in $f(x, y)$ that is approximately given by $\dfrac{\partial f}{\partial x}$. An analogous interpretation holds for $\dfrac{\partial f}{\partial y}$.

EXAMPLE 5 **Interpreting Partial Derivatives** Interpret the partial derivatives of $f(x, y) = 3x^2 + 2xy + 5y$ calculated in Example 4.

SOLUTION We showed in Example 4 that

$$\frac{\partial f}{\partial x}(1, 4) = 14, \qquad \frac{\partial f}{\partial y}(1, 4) = 7.$$

The fact that

$$\frac{\partial f}{\partial x}(1, 4) = 14$$

means that if y is kept constant at 4 and x is allowed to vary near 1, then $f(x, y)$ changes at a rate 14 times the change in x. That is, if x increases by one small unit, $f(x, y)$ increases by approximately 14 units. If x increases by h units (where h is small), $f(x, y)$ increases by approximately $14 \cdot h$ units. That is,

$$f(1 + h, 4) - f(1, 4) \approx 14 \cdot h.$$

Similarly, the fact that

$$\frac{\partial f}{\partial y}(1, 4) = 7$$

means that, if we keep x constant at 1 and let y vary near 4, then $f(x, y)$ changes at a rate equal to seven times the change in y. So, for a small value of k, we have

$$f(1, 4 + k) - f(1, 4) \approx 7 \cdot k.$$

▶ *Now Try Exercise 21*

We can generalize the interpretations of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ given in Example 5 to yield the following general fact:

Let $f(x, y)$ be a function of two variables. Then, if h and k are small, we have

$$f(a+h, b) - f(a, b) \approx \frac{\partial f}{\partial x}(a, b) \cdot h,$$

$$f(a, b+k) - f(a, b) \approx \frac{\partial f}{\partial y}(a, b) \cdot k.$$

Partial derivatives can be computed for functions of any number of variables. When taking the partial derivative with respect to one variable, we treat the other variables as constant.

EXAMPLE 6 **Partial Derivatives** Let $f(x, y, z) = x^2yz - 3z$.

(a) Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$. (b) Calculate $\frac{\partial f}{\partial z}(2, 3, 1)$.

SOLUTION (a) $\frac{\partial f}{\partial x} = 2xyz$, $\frac{\partial f}{\partial y} = x^2z$, $\frac{\partial f}{\partial z} = x^2y - 3$

(b) $\frac{\partial f}{\partial z}(2, 3, 1) = 2^2 \cdot 3 - 3 = 12 - 3 = 9$

▶ Now Try Exercise 15

EXAMPLE 7 **Heat-Loss Function** Let $f(x, y, z)$ be the heat-loss function computed in Example 2 of Section 1. That is, $f(x, y, z) = 11xy + 14yz + 15xz$. Calculate and interpret $\frac{\partial f}{\partial x}(10, 7, 5)$.

SOLUTION We have

$$\frac{\partial f}{\partial x} = 11y + 15z$$

$$\frac{\partial f}{\partial x}(10, 7, 5) = 11 \cdot 7 + 15 \cdot 5 = 77 + 75 = 152.$$

The quantity $\frac{\partial f}{\partial x}$ is commonly referred to as the *marginal heat loss with respect to change in* x. Specifically, if x is changed from 10 by h units (where h is small) and the values of y and z remain fixed at 7 and 5, the amount of heat loss will change by approximately $152 \cdot h$ units.

▶ Now Try Exercise 31

EXAMPLE 8 **Marginal Productivity of Capital** Consider the production function $f(x, y) = 60x^{3/4}y^{1/4}$, which gives the number of units of goods produced when x units of labor and y units of capital are used.

(a) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(b) Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x = 81$, $y = 16$.

(c) Interpret the numbers computed in part (b).

SOLUTION (a) $\frac{\partial f}{\partial x} = 60 \cdot \frac{3}{4}x^{-1/4}y^{1/4} = 45x^{-1/4}y^{1/4} = 45\frac{y^{1/4}}{x^{1/4}}$

$\frac{\partial f}{\partial y} = 60 \cdot \frac{1}{4}x^{3/4}y^{-3/4} = 15x^{3/4}y^{-3/4} = 15\frac{x^{3/4}}{y^{3/4}}$

(b) $\frac{\partial f}{\partial x}(81, 16) = 45 \cdot \frac{16^{1/4}}{81^{1/4}} = 45 \cdot \frac{2}{3} = 30$

$\frac{\partial f}{\partial y}(81, 16) = 15 \cdot \frac{81^{3/4}}{16^{3/4}} = 15 \cdot \frac{27}{8} = \frac{405}{8} = 50\frac{5}{8}$

(c) The quantities $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are referred to as the *marginal productivity of labor* and the *marginal productivity of capital.* If the amount of capital is held fixed at $y = 16$ and the amount of labor increases by 1 unit from 81, the quantity of goods produced will increase by approximately 30 units. Similarly, an increase in capital of 1 unit (with labor fixed at 81) results in an increase in production of approximately $50\frac{5}{8}$ units of goods.

▶ **Now Try Exercise 25**

Just as we formed second derivatives and derivatives of higher order in the case of one variable, we can form second partial derivatives and derivatives of higher order of a function $f(x, y)$ of two variables. Since $\frac{\partial f}{\partial x}$ is a function of x and y, we can differentiate it with respect to x or y. The partial derivative of $\frac{\partial f}{\partial x}$ with respect to x is denoted by $\frac{\partial^2 f}{\partial x^2}$. The partial derivative of $\frac{\partial f}{\partial x}$ with respect to y is denoted by $\frac{\partial^2 f}{\partial y\, \partial x}$. Similarly, the partial derivative of the function $\frac{\partial f}{\partial y}$ with respect to x is denoted by $\frac{\partial^2 f}{\partial x\, \partial y}$, and the partial derivative of $\frac{\partial f}{\partial y}$ with respect to y is denoted by $\frac{\partial^2 f}{\partial y^2}$. Almost all functions $f(x, y)$ encountered in applications [and all functions $f(x, y)$ in this text] have the property that

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y}.$$

When computing $\frac{\partial^2 f}{\partial y\, \partial x}$ and $\frac{\partial^2 f}{\partial x\, \partial y}$, note that verifying the last equation is a check that you have done the differentiation correctly.

EXAMPLE 9 **Partial Derivatives of Higher Order** Let $f(x, y) = x^2 + 3xy + 2y^2$. Calculate

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x\, \partial y}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y\, \partial x}.$$

SOLUTION First, we compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = 2x + 3y, \qquad \frac{\partial f}{\partial y} = 3x + 4y$$

To compute $\frac{\partial^2 f}{\partial x^2}$, we differentiate $\frac{\partial f}{\partial x}$ with respect to x:

$$\frac{\partial^2 f}{\partial x^2} = 2.$$

Similarly, to compute $\frac{\partial^2 f}{\partial y^2}$, we differentiate $\frac{\partial f}{\partial y}$ with respect to y:

$$\frac{\partial^2 f}{\partial y^2} = 4.$$

To compute $\frac{\partial^2 f}{\partial x\, \partial y}$, we differentiate $\frac{\partial f}{\partial y}$ with respect to x:

$$\frac{\partial^2 f}{\partial x\, \partial y} = 3.$$

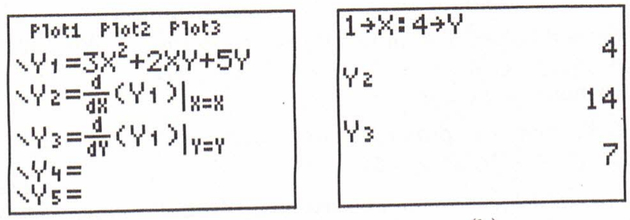
Finally, to compute $\frac{\partial^2 f}{\partial y\, \partial x}$, we differentiate $\frac{\partial f}{\partial x}$ with respect to y:

$$\frac{\partial^2 f}{\partial y\, \partial x} = 3.$$

▶ **Now Try Exercise 23**

INCORPORATING TECHNOLOGY

Evaluating Partial Derivatives The function from Example 4 and its first partial derivatives are specified in Fig. 3(a) and evaluated in Fig. 3(b). Recall that the expression $1 \to X$ is entered with [1] [STO ▷] [X,T,θ,n] and indicates that we are setting $X = 1$. The expression $4 \to Y$ has a similar meaning, but the variable Y is entered by means of [ALPHA] [Y]. We can also evaluate other partial derivatives. For example, we can find the partial derivative $\frac{\partial^2 f}{\partial x\, \partial y}$ in this case by setting **Y_4 = nDeriv(Y_3, X, X)**.



(a) (b)

Figure 3

Check Your Understanding 2

1. The number of TV sets an appliance store sells per week is given by a function of two variables, $f(x, y)$, where x is the price per TV set and y is the amount of money spent weekly on advertising. Suppose that the current price is \$400 per set and that currently \$2000 per week is being spent for advertising.

   (a) Would you expect $\frac{\partial f}{\partial x}(400, 2000)$ to be positive or negative?

   (b) Would you expect $\frac{\partial f}{\partial y}(400, 2000)$ to be positive or negative?

2. The monthly mortgage payment for a house is a function of two variables, $f(A, r)$, where A is the amount of the mortgage and the interest rate is r%. For a 30-year mortgage, $f(92{,}000, 3.5) = 412.78$ and $\frac{\partial f}{\partial r}(92{,}000, 3.5) = 38.47$. What is the significance of the number 38.47?

EXERCISES 2

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for each of the following functions.

1. $f(x, y) = 5xy$
2. $f(x, y) = x^2 - y^2$
3. $f(x, y) = 2x^2 e^y$
4. $f(x, y) = xe^{xy}$
5. $f(x, y) = \frac{x}{y} + \frac{y}{x}$
6. $f(x, y) = \frac{1}{x + y}$
7. $f(x, y) = (2x - y + 5)^2$
8. $f(x, y) = \frac{e^x}{1 + e^y}$
9. $f(x, y) = x^2 e^{3x} \ln y$
10. $f(x, y) = \ln(xy)$
11. $f(x, y) = \frac{x - y}{x + y}$
12. $f(x, y) = \sqrt{x^2 + y^2}$
13. Let $f(L, K) = 3\sqrt{LK}$. Find $\frac{\partial f}{\partial L}$.
14. Let $f(p, q) = 1 - p(1 + q)$. Find $\frac{\partial f}{\partial q}$ and $\frac{\partial f}{\partial p}$.
15. Let $f(x, y, z) = (1 + x^2 y)/z$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.
16. Let $f(x, y, z) = ze^{x/y}$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.
17. Let $f(x, y, z) = xze^{yz}$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.
18. Let $f(x, y, z) = \frac{xy}{z}$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.
19. Let $f(x, y) = x^2 + 2xy + y^2 + 3x + 5y$. Find $\frac{\partial f}{\partial x}(2, -3)$ and $\frac{\partial f}{\partial y}(2, -3)$.
20. Let $f(x, y) = (x + y^2)^3$. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(x, y) = (1, 2)$.
21. Let $f(x, y) = xy^2 + 5$. Evaluate $\frac{\partial f}{\partial y}$ at $(x, y) = (2, -1)$ and interpret your result.
22. Let $f(x, y) = \frac{x}{y - 6}$. Compute $\frac{\partial f}{\partial y}(2, 1)$ and interpret your result.
23. Let $f(x, y) = x^3 y + 2xy^2$. Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x\, \partial y}$, and $\frac{\partial^2 f}{\partial y\, \partial x}$.
24. Let $f(x, y) = xe^y + x^4 y + y^3$. Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x\, \partial y}$, and $\frac{\partial^2 f}{\partial y\, \partial x}$.
25. **Production** A farmer can produce $f(x, y) = 200\sqrt{6x^2 + y^2}$ units of produce by utilizing x units of labor and y units of capital. (The capital is used to rent or purchase land, materials, and equipment.)

(a) Calculate the marginal productivities of labor and capital when $x = 10$ and $y = 5$.

(b) Let h be a small number. Use the result of part (a) to determine the approximate effect on production of changing labor from 10 to $10 + h$ units while keeping capital fixed at 5 units.

(c) Use part (b) to estimate the change in production when labor decreases from 10 to 9.5 units and capital stays fixed at 5 units.

26. **Productivity Labor and Capital** The productivity of a country is given by $f(x, y) = 300x^{2/3}y^{1/3}$, where x and y are the amount of labor and capital.

(a) Compute the marginal productivities of labor and capital when $x = 125$ and $y = 64$.

(b) Use part (a) to determine the approximate effect on productivity of increasing capital from 64 to 66 units, while keeping labor fixed at 125 units.

(c) What would be the approximate effect of decreasing labor from 125 to 124 units while keeping capital fixed at 64 units?

27. **Modes of Transportation** In a certain suburban community, commuters have the choice of getting into the city by bus or train. The demand for these modes of transportation varies with their cost. Let $f(p_1, p_2)$ be the number of people who will take the bus when p_1 is the price of the bus ride and p_2 is the price of the train ride. For example, if $f(4.50, 6) = 7000$, then 7000 commuters will take the bus when the price of a bus ticket is \$4.50 and the price of a train ticket is \$6.00. Explain why $\frac{\partial f}{\partial p_1} < 0$ and $\frac{\partial f}{\partial p_2} > 0$.

28. Refer to Exercise 27. Let $g(p_1, p_2)$ be the number of people who will take the train when p_1 is the price of the bus ride and p_2 is the price of the train ride. Would you expect $\frac{\partial g}{\partial p_1}$ to be positive or negative? How about $\frac{\partial g}{\partial p_2}$?

29. Let p_1 be the average price of MP3 players, p_2 the average price of audio files, $f(p_1, p_2)$ the demand for MP3 players, and $g(p_1, p_2)$ the demand for audio files. Explain why $\frac{\partial f}{\partial p_2} < 0$ and $\frac{\partial g}{\partial p_1} < 0$.

30. The demand for a certain gas-guzzling car is given by $f(p_1, p_2)$, where p_1 is the price of the car and p_2 is the price of gasoline. Explain why $\frac{\partial f}{\partial p_1} < 0$ and $\frac{\partial f}{\partial p_2} < 0$.

31. The volume (V) of a certain amount of a gas is determined by the temperature (T) and the pressure (P) by the formula $V = .08(T/P)$. Calculate and interpret $\frac{\partial V}{\partial P}$ and $\frac{\partial V}{\partial T}$ when $P = 20$, $T = 300$.

32. **Beer Consumption** Using data collected from 1929 to 1941, Richard Stone determined that the yearly quantity Q of beer consumed in the United Kingdom was approximately given by the formula $Q = f(m, p, r, s)$, where

$$f(m, p, r, s) = (1.058)m^{.136}p^{-.727}r^{.914}s^{.816}$$

and m is the aggregate real income (personal income after direct taxes, adjusted for retail price changes), p is the average retail price of the commodity (in this case, beer), r is the average retail price level of all other consumer goods and services, and s is a measure of the strength of the beer. Determine which partial derivatives are positive and which are negative, and give interpretations. (For example, since $\frac{\partial f}{\partial r} > 0$, people buy more beer when the prices of other goods increase and the other factors remain constant.) (*Source: Journal of the Royal Statistical Society.*)

33. Richard Stone (see Exercise 32) determined that the yearly consumption of food in the United States was given by

$$f(m, p, r) = (2.186)m^{.595}p^{-.543}r^{.922}.$$

Determine which partial derivatives are positive and which are negative, and give interpretations of these facts.

34. **Distribution of Revenue** For the production function $f(x, y) = 60x^{3/4}y^{1/4}$ considered in Example 8, think of $f(x, y)$ as the revenue when x units of labor and y units of capital are used. Under actual operating conditions—say, $x = a$ and $y = b$—$\frac{\partial f}{\partial x}(a, b)$ is referred to as the *wage per unit of labor* and $\frac{\partial f}{\partial y}(a, b)$ is referred to as the *wage per unit of capital.* Show that

$$f(a, b) = a \cdot \left[\frac{\partial f}{\partial x}(a, b)\right] + b \cdot \left[\frac{\partial f}{\partial y}(a, b)\right].$$

(This equation shows how the revenue is distributed between labor and capital.)

35. Compute $\frac{\partial^2 f}{\partial x^2}$, where $f(x, y) = 60x^{3/4}y^{1/4}$, a production function (where x is units of labor). Explain why $\frac{\partial^2 f}{\partial x^2}$ is always negative.

36. Compute $\frac{\partial^2 f}{\partial y^2}$, where $f(x, y) = 60x^{3/4}y^{1/4}$, a production function (where y is units of capital). Explain why $\frac{\partial^2 f}{\partial y^2}$ is always negative.

37. Let $f(x, y) = 3x^2 + 2xy + 5y$, as in Example 5. Show that

$$f(1 + h, 4) - f(1, 4) = 14h + 3h^2.$$

Thus, the error in approximating $f(1 + h, 4) - f(1, 4)$ by $14h$ is $3h^2$. (If $h = .01$, for instance, the error is only .0003.)

38. **Body Surface Area** Physicians, particularly pediatricians, sometimes need to know the body surface area of a patient. For instance, they use surface area to adjust the results of certain tests of kidney performance. Tables are available that give the approximate body surface area A in square meters of a person who weighs W kilograms and is H centimeters tall. The following empirical formula is also used:

$$A = .007W^{.425}H^{.725}.$$

Evaluate $\frac{\partial A}{\partial W}$ and $\frac{\partial A}{\partial H}$ when $W = 54$ and $H = 165$, and give a physical interpretation of your answers. You may use the approximations $(54)^{.425} \approx 5.4$, $(54)^{-.575} \approx .10$, $(165)^{.725} \approx 40.5$, and $(165)^{-.275} \approx .25$. (*Source: Mathematical Preparation for Laboratory Technicians.*)

Solutions to Check Your Understanding 2

1. (a) Negative. $\frac{\partial f}{\partial x}(400, 2000)$ is approximately the change in sales due to a \$1 increase in x (price). Since raising prices lowers sales, we would expect $\frac{\partial f}{\partial x}(400, 2000)$ to be negative.

   (b) Positive. $\frac{\partial f}{\partial y}(400, 2000)$ is approximately the change in sales due to a \$1 increase in advertising. Since spending more money on advertising brings in more customers, we would expect sales to increase; that is, $\frac{\partial f}{\partial y}(400, 2000)$ is most likely positive.

2. If the interest rate is raised from 3.5% to 4.5%, the monthly payment will increase by about \$38.67. [An increase to 4% causes an increase in the monthly payment of about $\frac{1}{2} \cdot (38.47)$ or \$19.24, and so on.]

3 Maxima and Minima of Functions of Several Variables

Previously, we studied how to determine the maxima and minima of functions of a single variable. Let us extend that discussion to functions of several variables.

If $f(x, y)$ is a function of two variables, we say that $f(x, y)$ has a *relative maximum* when $x = a$, $y = b$ if $f(x, y)$ is at most equal to $f(a, b)$ whenever x is near a and y is near b. Geometrically, the graph of $f(x, y)$ has a peak at $(x, y) = (a, b)$. [See Fig. 1(a).] Similarly, we say that $f(x, y)$ has a *relative minimum* when $x = a$, $y = b$, if $f(x, y)$ is at least equal to $f(a, b)$ whenever x is near a and y is near b. Geometrically, the graph of $f(x, y)$ has a pit whose bottom occurs at $(x, y) = (a, b)$. [See Fig. 1(b).]

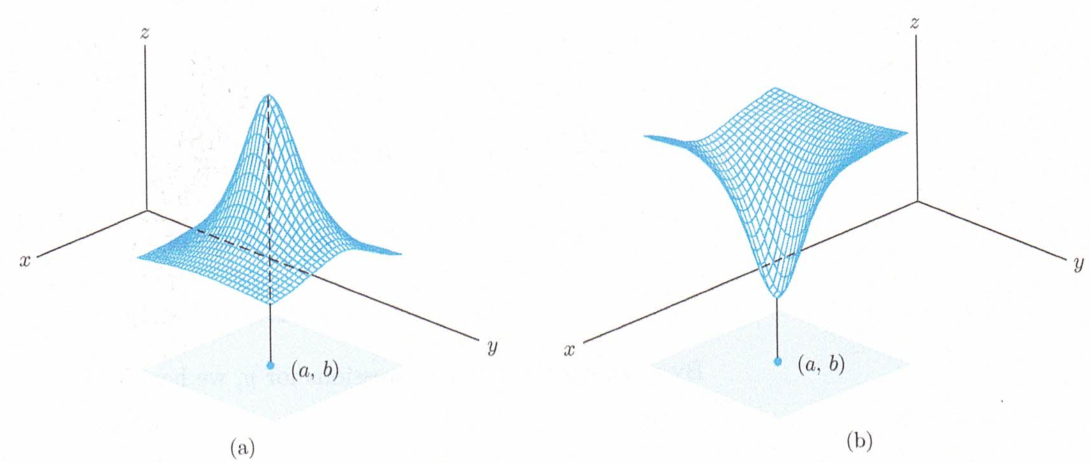


Figure 1 Maximum and minimum points.

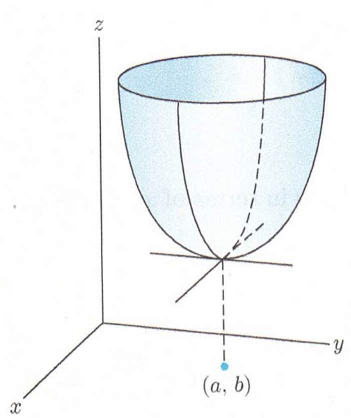


Figure 2 Horizontal tangent lines at a relative minimum.

Suppose that the function $f(x, y)$ has a relative minimum at $(x, y) = (a, b)$, as in Fig. 2. When y is held constant at b, $f(x, y)$ is a function of x with a relative minimum at $x = a$. Therefore, the tangent line to the curve $z = f(x, b)$ is horizontal at $x = a$ and hence has slope 0. That is,

$$\frac{\partial f}{\partial x}(a, b) = 0.$$

Likewise, when x is held constant at a, then $f(x, y)$ is a function of y with a relative minimum at $y = b$. Therefore, its derivative with respect to y is zero at $y = b$. That is,

$$\frac{\partial f}{\partial y}(a, b) = 0.$$

Similar considerations apply when $f(x, y)$ has a relative maximum at $(x, y) = (a, b)$.

First-Derivative Test for Functions of Two Variables If $f(x, y)$ has either a relative maximum or minimum at $(x, y) = (a, b)$, then

$$\frac{\partial f}{\partial x}(a, b) = 0$$

and

$$\frac{\partial f}{\partial y}(a, b) = 0.$$

A relative maximum or minimum may or may not be an absolute maximum or minimum. However, to simplify matters in this text, the examples and exercises have been chosen so that, if an absolute extremum of $f(x, y)$ exists, it will occur at a point where $f(x, y)$ has a relative extremum.

EXAMPLE 1 **Locating a Minimum Value** The function $f(x, y) = 3x^2 - 4xy + 3y^2 + 8x - 17y + 30$ has the graph pictured in Fig. 2. Find the point (a, b) at which $f(x, y)$ attains its minimum value.

SOLUTION We look for those values of x and y at which both partial derivatives are zero. The partial derivatives are

$$\frac{\partial f}{\partial x} = 6x - 4y + 8,$$

$$\frac{\partial f}{\partial y} = -4x + 6y - 17.$$

Setting $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, we obtain

$$6x - 4y + 8 = 0 \quad \text{or} \quad y = \frac{6x + 8}{4},$$

$$-4x + 6y - 17 = 0 \quad \text{or} \quad y = \frac{4x + 17}{6}.$$

By equating these two expressions for y, we have

$$\frac{6x + 8}{4} = \frac{4x + 17}{6}.$$

Cross-multiplying, we see that

$$36x + 48 = 16x + 68$$

$$20x = 20$$

$$x = 1.$$

When we substitute this value for x in our first equation for y in terms of x, we obtain

$$y = \frac{6x + 8}{4} = \frac{6 \cdot 1 + 8}{4} = \frac{7}{2}.$$

If $f(x, y)$ has a minimum, it must occur where $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. We have determined that the partial derivatives are zero only when $x = 1$, $y = \frac{7}{2}$. From Fig. 2 we know that $f(x, y)$ has a minimum, so it must be at $(x, y) = (1, \frac{7}{2})$.

▶ **Now Try Exercise 3**

EXAMPLE 2 **Price Discrimination** A firm markets a product in two countries and can charge different amounts in each country. Let x be the number of units to be sold in the first country and y the number of units to be sold in the second country. Due to the laws of demand, the firm must set the price at $97 - (x/10)$ dollars in the first country and $83 - (y/20)$ dollars in the second country to sell all the units. The cost of producing these units is $20{,}000 + 3(x + y)$. Find the values of x and y that maximize the profit.

SOLUTION Let $f(x, y)$ be the profit derived from selling x units in the first country and y in the second. Then,

$$\begin{aligned} f(x,y) &= [\text{revenue from first country}] + [\text{revenue from second country}] - [\text{cost}] \\ &= \left(97 - \frac{x}{10}\right)x + \left(83 - \frac{y}{20}\right)y - [20{,}000 + 3(x+y)] \\ &= 97x - \frac{x^2}{10} + 83y - \frac{y^2}{20} - 20{,}000 - 3x - 3y \\ &= 94x - \frac{x^2}{10} + 80y - \frac{y^2}{20} - 20{,}000. \end{aligned}$$

To find where $f(x, y)$ has its maximum value, we look for those values of x and y at which both partial derivatives are zero.

$$\frac{\partial f}{\partial x} = 94 - \frac{x}{5}$$

$$\frac{\partial f}{\partial y} = 80 - \frac{y}{10}$$

We set $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ to obtain

$$94 - \frac{x}{5} = 0 \quad \text{or} \quad x = 470,$$

$$80 - \frac{y}{10} = 0 \quad \text{or} \quad y = 800.$$

Therefore, the firm should adjust its prices to levels where it will sell 470 units in the first country and 800 units in the second country. ▶ ***Now Try Exercise 9***

EXAMPLE 3 **Heat Loss** Suppose that we want to design a rectangular building having a volume of 147,840 cubic feet. Assuming that the daily loss of heat is given by

$$w = 11xy + 14yz + 15xz,$$

where x, y, and z are, respectively, the length, width, and height of the building, find the dimensions of the building for which the daily heat loss is minimal.

SOLUTION We must minimize the function

$$w = 11xy + 14yz + 15xz, \tag{1}$$

where x, y, z satisfy the constraint equation

$$xyz = 147{,}840.$$

For simplicity, let us denote 147,840 by V. Then, $xyz = V$, so $z = V/xy$. We substitute this expression for z in the objective function (1) to obtain a heat-loss function $g(x, y)$ of two variables:

$$g(x, y) = 11xy + 14y\frac{V}{xy} + 15x\frac{V}{xy} = 11xy + \frac{14V}{x} + \frac{15V}{y}.$$

To minimize this function, we first compute the partial derivatives with respect to x and y; then we equate them to zero.

$$\frac{\partial g}{\partial x} = 11y - \frac{14V}{x^2} = 0$$

$$\frac{\partial g}{\partial y} = 11x - \frac{15V}{y^2} = 0$$

These two equations yield

$$y = \frac{14V}{11x^2}, \qquad (2)$$

$$11xy^2 = 15V. \qquad (3)$$

If we substitute the value of y from (2) in (3), we see that

$$11x\left(\frac{14V}{11x^2}\right)^2 = 15V$$

$$\frac{14^2V^2}{11x^3} = 15V$$

$$x^3 = \frac{14^2 \cdot V^2}{11 \cdot 15 \cdot V} = \frac{14^2 \cdot V}{11 \cdot 15}$$

$$= \frac{14^2 \cdot 147{,}840}{11 \cdot 15}$$

$$= 175{,}616.$$

Therefore, we see (using a calculator) that

$$x = 56.$$

From equation (2) we find that

$$y = \frac{14 \cdot V}{11x^2} = \frac{14 \cdot 147{,}840}{11 \cdot 56^2} = 60.$$

Finally,

$$z = \frac{V}{xy} = \frac{147{,}840}{56 \cdot 60} = 44.$$

Thus, the building should be 56 feet long, 60 feet wide, and 44 feet high to minimize the heat loss. For further discussion of this heat-loss problem, as well as other examples of optimization in architectural design, see *Urban Space and Structure*.

▶ **Now Try Exercise 27**

When considering a function of two variables, we find points (x, y) at which $f(x, y)$ has a potential relative maximum or minimum by setting $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ equal to zero and solving for x and y. However, if we are given no additional information about $f(x, y)$, it may be difficult to determine whether we have found a maximum or a minimum (or neither). In the case of functions of one variable, we studied concavity and deduced the second-derivative test. There is an analog of the second-derivative test for functions of two variables, but it is much more complicated than the one-variable test. We state it without proof.

Second-Derivative Test for Functions of Two Variables Suppose that $f(x, y)$ is a function and (a, b) is a point at which

$$\frac{\partial f}{\partial x}(a, b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b) = 0,$$

and let

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2.$$

1. If

$$D(a, b) > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(a, b) > 0,$$

then $f(x, y)$ has a relative minimum at (a, b).

2. If

$$D(a, b) > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(a, b) < 0,$$

then $f(x, y)$ has a relative maximum at (a, b).

3. If

$$D(a, b) < 0,$$

then $f(x, y)$ has neither a relative maximum nor a relative minimum at (a, b).

4. If $D(a, b) = 0$, no conclusion can be drawn from this test.

The saddle-shaped graph in Fig. 3 illustrates a function $f(x, y)$ for which $D(a, b) < 0$. Both partial derivatives are zero at $(x, y) = (a, b)$, and yet the function has neither a relative maximum nor a relative minimum there. (Observe that the function has a relative maximum with respect to x when y is held constant and a relative minimum with respect to y when x is held constant.)

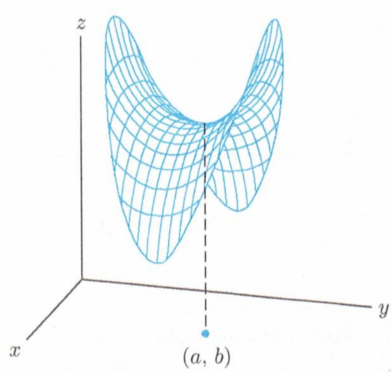


Figure 3

EXAMPLE 4 **Applying the Second Derivative Test** Let $f(x, y) = x^3 - y^2 - 12x + 6y + 5$. Find all possible relative maximum and minimum points of $f(x, y)$. Use the second-derivative test to determine the nature of each such point.

SOLUTION Since

$$\frac{\partial f}{\partial x} = 3x^2 - 12, \qquad \frac{\partial f}{\partial y} = -2y + 6,$$

we find that $f(x, y)$ has a potential relative extreme point when

$$3x^2 - 12 = 0,$$
$$-2y + 6 = 0.$$

From the first equation, $3x^2 = 12$, $x^2 = 4$, and $x = \pm 2$. From the second equation, $y = 3$. Thus, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero when $(x, y) = (2, 3)$ and when $(x, y) = (-2, 3)$.

To apply the second-derivative test, compute

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial y^2} = -2, \qquad \frac{\partial^2 f}{\partial x\,\partial y} = 0,$$

and

$$D(x,y) = \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 = (6x)(-2) - 0^2 = -12x. \tag{4}$$

Since $D(2,3) = -12(2) = -24$, which is negative, case 3 of the second-derivative test says that $f(x,y)$ has neither a relative maximum nor a relative minimum at $(2,3)$. However, $D(-2,3) = -12(-2) = 24$. Since $D(-2,3)$ is positive, the function $f(x,y)$ has either a relative maximum or a relative minimum at $(-2,3)$. To determine which, we compute

$$\frac{\partial^2 f}{\partial x^2}(-2,3) = 6(-2) = -12 < 0.$$

By case 2 of the second-derivative test, the function $f(x,y)$ has a relative maximum at $(-2,3)$.

▶ *Now Try Exercise 19*

In this section we have restricted ourselves to functions of two variables, but the case of three or more variables is handled in a similar fashion. For instance, here is the first-derivative test for a function of three variables.

If $f(x,y,z)$ has a relative maximum or minimum at $(x,y,z) = (a,b,c)$, then

$$\frac{\partial f}{\partial x}(a,b,c) = 0,$$

$$\frac{\partial f}{\partial y}(a,b,c) = 0,$$

$$\frac{\partial f}{\partial z}(a,b,c) = 0.$$

Check Your Understanding 3

1. Find all points (x,y) where $f(x,y) = x^3 - 3xy + \frac{1}{2}y^2 + 8$ has a possible relative maximum or minimum.
2. Apply the second-derivative test to the function $g(x,y)$ of Example 3 to confirm that a relative minimum actually occurs when $x = 56$ and $y = 60$.

EXERCISES 3

Find all points (x,y) where $f(x,y)$ has a possible relative maximum or minimum.

1. $f(x,y) = x^2 - 3y^2 + 4x + 6y + 8$
2. $f(x,y) = \frac{1}{2}x^2 + y^2 - 3x + 2y - 5$
3. $f(x,y) = x^2 - 5xy + 6y^2 + 3x - 2y + 4$
4. $f(x,y) = -3x^2 + 7xy - 4y^2 + x + y$
5. $f(x,y) = x^3 + y^2 - 3x + 6y$
6. $f(x,y) = x^2 - y^3 + 5x + 12y + 1$
7. $f(x,y) = \frac{1}{3}x^3 - 2y^3 - 5x + 6y - 5$
8. $f(x,y) = x^4 - 8xy + 2y^2 - 3$
9. The function $f(x,y) = 2x + 3y + 9 - x^2 - xy - y^2$ has a maximum at some point (x,y). Find the values of x and y where this maximum occurs.
10. The function $f(x,y) = \frac{1}{2}x^2 + 2xy + 3y^2 - x + 2y$ has a minimum at some point (x,y). Find the values of x and y where this minimum occurs.

In Exercises 11–16, both first partial derivatives of the function $f(x,y)$ are zero at the given points. Use the second-derivative test to determine the nature of $f(x,y)$ at each of these points. If the second-derivative test is inconclusive, so state.

11. $f(x,y) = 3x^2 - 6xy + y^3 - 9y$; $(3,3)$, $(-1,-1)$
12. $f(x,y) = 6xy^2 - 2x^3 - 3y^4$; $(0,0)$, $(1,1)$, $(1,-1)$
13. $f(x,y) = 2x^2 - x^4 - y^2$; $(-1,0)$, $(0,0)$, $(1,0)$
14. $f(x,y) = x^4 - 4xy + y^4$; $(0,0)$, $(1,1)$, $(-1,-1)$
15. $f(x,y) = ye^x - 3x - y + 5$; $(0,3)$
16. $f(x,y) = \dfrac{1}{x} + \dfrac{1}{y} + xy$; $(1,1)$

Find all points (x,y) where $f(x,y)$ has a possible relative maximum or minimum. Then, use the second-derivative test to determine, if possible, the nature of $f(x,y)$ at each of these points. If the second-derivative test is inconclusive, so state.

17. $f(x,y) = x^2 - 2xy + 4y^2$
18. $f(x,y) = 2x^2 + 3xy + 5y^2$

19. $f(x, y) = -2x^2 + 2xy - y^2 + 4x - 6y + 5$

20. $f(x, y) = -x^2 - 8xy - y^2$

21. $f(x, y) = x^2 + 2xy + 5y^2 + 2x + 10y - 3$

22. $f(x, y) = x^2 - 2xy + 3y^2 + 4x - 16y + 22$

23. $f(x, y) = x^3 - y^2 - 3x + 4y$

24. $f(x, y) = x^3 - 2xy + 4y$

25. $f(x, y) = 2x^2 + y^3 - x - 12y + 7$

26. $f(x, y) = x^2 + 4xy + 2y^4$

27. Find the possible values of x, y, z at which

$$f(x, y, z) = 2x^2 + 3y^2 + z^2 - 2x - y - z$$

assumes its minimum value.

28. Find the possible values of x, y, z at which

$$f(x, y, z) = 5 + 8x - 4y + x^2 + y^2 + z^2$$

assumes its minimum value.

29. **Maximizing Volume** U.S. postal rules require that the length plus the girth of a package cannot exceed 84 inches. Find the dimensions of the rectangular package of greatest volume that can be mailed. [*Note:* From Fig. 4 we see that $84 = (\text{length}) + (\text{girth}) = l + (2x + 2y)$.]

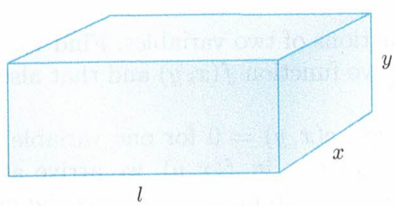


Figure 4

30. **Minimizing Surface Area** Find the dimensions of the rectangular box of least surface area that has a volume of 1000 cubic inches.

31. **Maximizing Profit** A company manufactures and sells two products, I and II, that sell for \$10 and \$9 per unit, respectively. The cost of producing x units of product I and y units of product II is

$$400 + 2x + 3y + .01(3x^2 + xy + 3y^2).$$

Find the values of x and y that maximize the company's profits. [*Note:* Profit = (revenue) − (cost).]

32. **Maximizing Profit** A monopolist manufactures and sells two competing products, I and II, that cost \$30 and \$20 per unit, respectively, to produce. The revenue from marketing x units of product I and y units of product II is $98x + 112y - .04xy - .1x^2 - .2y^2$. Find the values of x and y that maximize the monopolist's profits.

33. **Profit from Two Products** A company manufactures and sells two products, I and II, that sell for $\$p_{\text{I}}$ and $\$p_{\text{II}}$ per unit, respectively. Let $C(x, y)$ be the cost of producing x units of product I and y units of product II. Show that if the company's profit is maximized when $x = a$, $y = b$, then

$$\frac{\partial C}{\partial x}(a, b) = p_{\text{I}} \quad \text{and} \quad \frac{\partial C}{\partial y}(a, b) = p_{\text{II}}.$$

34. **Revenue from Two Products** A company manufactures and sells two competing products, I and II, that cost $\$p_{\text{I}}$ and $\$p_{\text{II}}$ per unit, respectively, to produce. Let $R(x, y)$ be the revenue from marketing x units of product I and y units of product II. Show that if the company's profit is maximized when $x = a$, $y = b$, then

$$\frac{\partial R}{\partial x}(a, b) = p_{\text{I}} \quad \text{and} \quad \frac{\partial R}{\partial y}(a, b) = p_{\text{II}}.$$

Solutions to Check Your Understanding 3

1. Compute the first partial derivatives of $f(x, y)$ and solve the system of equations that results from setting the partials equal to zero.

$$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0$$

$$\frac{\partial f}{\partial y} = -3x + y = 0$$

Solve each equation for y in terms of x.

$$\begin{cases} y = x^2 \\ y = 3x \end{cases}$$

Equate expressions for y and solve for x.

$$\begin{aligned} x^2 &= 3x \\ x^2 - 3x &= 0 \\ x(x - 3) &= 0 \\ x = 0 \quad &\text{or} \quad x = 3 \end{aligned}$$

When $x = 0$, $y = 0^2 = 0$. When $x = 3$, $y = 3^2 = 9$. Therefore, the possible relative maximum or minimum points are $(0, 0)$ and $(3, 9)$.

2. We have

$$g(x, y) = 11xy + \frac{14V}{x} + \frac{15V}{y},$$

$$\frac{\partial g}{\partial x} = 11y - \frac{14V}{x^2}, \quad \text{and} \quad \frac{\partial g}{\partial y} = 11x - \frac{15V}{y^2}.$$

Now,

$$\frac{\partial^2 g}{\partial x^2} = \frac{28V}{x^3}, \quad \frac{\partial^2 g}{\partial y^2} = \frac{30V}{y^3}, \quad \text{and} \quad \frac{\partial^2 g}{\partial x\, \partial y} = 11.$$

Therefore,

$$D(x, y) = \frac{28V}{x^3} \cdot \frac{30V}{y^3} - (11)^2,$$

$$\begin{aligned} D(56, 60) &= \frac{28(147{,}840)}{(56)^3} \cdot \frac{30(147{,}840)}{(60)^3} - 121, \\ &= 484 - 121 = 363 > 0, \end{aligned}$$

and

$$\frac{\partial^2 g}{\partial x^2}(56, 60) = \frac{28(147{,}840)}{(56)^3} > 0.$$

It follows that $g(x, y)$ has a relative minimum at $x = 56$, $y = 60$.

4 Lagrange Multipliers and Constrained Optimization

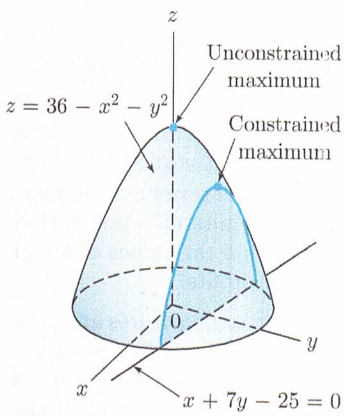


Figure 1 A constrained optimization problem.

We have seen a number of optimization problems in which we were required to minimize (or maximize) an objective function where the variables were subject to a constraint equation. For instance, in one problem, we minimized the cost of a rectangular enclosure by minimizing the objective function $42x + 28y$, where x and y were subject to the constraint equation $600 - xy = 0$. In the preceding section (Example 3), we minimized the daily heat loss from a building by minimizing the objective function $11xy + 14yz + 15xz$, subject to the constraint equation $147{,}840 - xyz = 0$.

Figure 1 gives a graphical illustration of what happens when an objective function is maximized subject to a constraint. The graph of the objective function is the cone-shaped surface $z = 36 - x^2 - y^2$, and the colored curve on that surface consists of those points whose x- and y-coordinates satisfy the constraint equation $x + 7y - 25 = 0$. The constrained maximum is at the highest point on this curve. Of course, the surface itself has a higher "unconstrained maximum" at $(x, y, z) = (0, 0, 36)$, but these values of x and y do not satisfy the constraint equation.

In this section, we introduce a powerful technique for solving problems of this type. Let us begin with the following general problem, which involves two variables.

Problem Let $f(x, y)$ and $g(x, y)$ be functions of two variables. Find values of x and y that maximize (or minimize) the objective function $f(x, y)$ and that also satisfy the constraint equation $g(x, y) = 0$.

Of course, if we can solve the equation $g(x, y) = 0$ for one variable in terms of the other and substitute the resulting expression in $f(x, y)$, we arrive at a function of a single variable that can be maximized (or minimized) by using the methods you may have previously learned. However, this technique can be unsatisfactory for two reasons. First, it may be difficult to solve the equation $g(x, y) = 0$ for x or for y. For example, if $g(x, y) = x^4 + 5x^3y + 7x^2y^3 + y^5 - 17 = 0$, it is difficult to write y as a function of x or x as a function of y. Second, even if $g(x, y) = 0$ can be solved for one variable in terms of the other, substitution of the result into $f(x, y)$ may yield a complicated function.

One clever idea for handling the preceding problem was discovered by the eighteenth-century mathematician Lagrange, and the technique that he pioneered today bears his name, the *method of Lagrange multipliers*. The basic idea of this method is to replace $f(x, y)$ by an auxiliary function of three variables $F(x, y, \lambda)$, defined as

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

The new variable λ (lambda) is called a *Lagrange multiplier* and always multiplies the constraint function $g(x, y)$. The following theorem is stated without proof.

Theorem Suppose that, subject to the constraint $g(x, y) = 0$, the function $f(x, y)$ has a relative maximum or minimum at $(x, y) = (a, b)$. Then, there is a value of λ—say, $\lambda = c$—such that the partial derivatives of $F(x, y, \lambda)$ all equal zero at $(x, y, \lambda) = (a, b, c)$.

The theorem implies that, if we locate all points (x, y, λ) where the partial derivatives of $F(x, y, \lambda)$ are all zero, among the corresponding points (x, y), we then will find all possible places where $f(x, y)$ may have a constrained relative maximum or minimum. Thus, the first step in the method of Lagrange multipliers is to set the partial derivatives of $F(x, y, \lambda)$ equal to zero and solve for x, y, and λ:

$$\frac{\partial F}{\partial x} = 0 \tag{L-1}$$

$$\frac{\partial F}{\partial y} = 0 \tag{L-2}$$

$$\frac{\partial F}{\partial \lambda} = 0. \tag{L-3}$$

From the definition of $F(x, y, \lambda)$, we see that $\frac{\partial F}{\partial \lambda} = g(x, y)$. Thus, the third equation (L-3) is just the original constraint equation $g(x, y) = 0$. So, when we find a point (x, y, λ) that satisfies (L-1), (L-2), and (L-3), the coordinates x and y will automatically satisfy the constraint equation.

The first example applies this method to the problem described in Fig. 1.

EXAMPLE 1 **Lagrange Multipliers** Maximize $36 - x^2 - y^2$ subject to the constraint $x + 7y - 25 = 0$.

SOLUTION Here, $f(x, y) = 36 - x^2 - y^2$, $g(x, y) = x + 7y - 25$, and

$$F(x, y, \lambda) = 36 - x^2 - y^2 + \lambda(x + 7y - 25).$$

Equations (L-1) to (L-3) read

$$\frac{\partial F}{\partial x} = -2x + \lambda = 0, \tag{1}$$

$$\frac{\partial F}{\partial y} = -2y + 7\lambda = 0, \tag{2}$$

$$\frac{\partial F}{\partial \lambda} = x + 7y - 25 = 0. \tag{3}$$

We solve the first two equations for λ:

$$\lambda = 2x$$

$$\lambda = \frac{2}{7}y. \tag{4}$$

If we equate these two expressions for λ, we obtain

$$2x = \frac{2}{7}y$$

$$x = \frac{1}{7}y. \tag{5}$$

Substituting this expression for x in equation (3), we have

$$\frac{1}{7}y + 7y - 25 = 0$$

$$\frac{50}{7}y = 25$$

$$y = \frac{7}{2}.$$

With this value for y, equations (4) and (5) produce the values of x and λ:

$$x = \frac{1}{7}y = \frac{1}{7}\left(\frac{7}{2}\right) = \frac{1}{2},$$

$$\lambda = \frac{2}{7}y = \frac{2}{7}\left(\frac{7}{2}\right) = 1.$$

Therefore, the partial derivatives of $F(x, y, \lambda)$ are zero when $x = \frac{1}{2}$, $y = \frac{7}{2}$, and $\lambda = 1$. So, the maximum value of $36 - x^2 - y^2$ subject to the constraint $x + 7y - 25 = 0$ is

$$36 - \left(\frac{1}{2}\right)^2 - \left(\frac{7}{2}\right)^2 = \frac{47}{2}.$$

▶ Now Try Exercise 1

The preceding technique for solving three equations in the three variables x, y, and λ can usually be applied to solve Lagrange multiplier problems. Here is the basic procedure:

1. Solve (L-1) and (L-2) for λ in terms of x and y; then, equate the resulting expressions for λ.
2. Solve the resulting equation for one of the variables.
3. Substitute the expression so derived in the equation (L-3), and solve the resulting equation of one variable.
4. Use the one known variable and the equations of steps 1 and 2 to determine the other two variables.

In most applications, we know that an absolute (constrained) maximum or minimum exists. In the event that the method of Lagrange multipliers produces exactly one possible relative extreme value, we will assume that it is indeed the sought-after absolute extreme value. For instance, the statement of Example 1 is meant to imply that there is an absolute maximum value. Since we determined that there was just one possible relative extreme value, we concluded that it was the absolute maximum value.

EXAMPLE 2 **Lagrange Multipliers** Using Lagrange multipliers, minimize $42x + 28y$, subject to the constraint $600 - xy = 0$, where x and y are restricted to positive values.

SOLUTION We have $f(x, y) = 42x + 28y$, $g(x, y) = 600 - xy$, and

$$F(x, y, \lambda) = 42x + 28y + \lambda(600 - xy).$$

The equations (L-1) to (L-3), in this case, are

$$\frac{\partial F}{\partial x} = 42 - \lambda y = 0,$$

$$\frac{\partial F}{\partial y} = 28 - \lambda x = 0,$$

$$\frac{\partial F}{\partial \lambda} = 600 - xy = 0.$$

From the first two equations, we see that

$$\lambda = \frac{42}{y} = \frac{28}{x}. \qquad \text{(step 1)}$$

Therefore,

$$42x = 28y$$

and

$$x = \frac{2}{3}y. \qquad \text{(step 2)}$$

Substituting this expression for x in the third equation, we derive

$$600 - \left(\frac{2}{3}y\right) y = 0$$

$$y^2 = \frac{3}{2} \cdot 600 = 900$$

$$y = \pm 30. \qquad \text{(step 3)}$$

We discard the case $y = -30$ because we are interested only in positive values of x and y. Using $y = 30$, we find that

$$\left.\begin{aligned} x &= \frac{2}{3}(30) = 20 \\ \lambda &= \frac{28}{20} = \frac{7}{5}. \end{aligned}\right\} \qquad \text{(step 4)}$$

So the minimum value of $42x + 28y$ with x and y subject to the constraint occurs when $x = 20$, $y = 30$, and $\lambda = \frac{7}{5}$. That minimum value is

$$42 \cdot (20) + 28 \cdot (30) = 1680.$$

▶ **Now Try Exercise 3**

EXAMPLE 3 **Maximizing Production** Suppose that x units of labor and y units of capital can produce $f(x, y) = 60x^{3/4}y^{1/4}$ units of a certain product. Also, suppose that each unit of labor costs \$100, whereas each unit of capital costs \$200. Assume that \$30,000 is available to spend on production. How many units of labor and how many units of capital should be utilized to maximize production?

SOLUTION The cost of x units of labor and y units of capital equals $100x + 200y$. Therefore, since we want to use all the available money (\$30,000), we must satisfy the constraint equation

$$100x + 200y = 30{,}000$$

or

$$g(x, y) = 30{,}000 - 100x - 200y = 0.$$

The objective function is $f(x, y) = 60x^{3/4}y^{1/4}$. In this case, we have

$$F(x, y, \lambda) = 60x^{3/4}y^{1/4} + \lambda(30{,}000 - 100x - 200y).$$

The equations (L-1) to (L-3) read

$$\frac{\partial F}{\partial x} = 45x^{-1/4}y^{1/4} - 100\lambda = 0, \qquad \text{(L-1)}$$

$$\frac{\partial F}{\partial y} = 15x^{3/4}y^{-3/4} - 200\lambda = 0, \qquad \text{(L-2)}$$

$$\frac{\partial F}{\partial \lambda} = 30{,}000 - 100x - 200y = 0. \qquad \text{(L-3)}$$

By solving the first two equations for λ, we see that

$$\lambda = \frac{45}{100}x^{-1/4}y^{1/4} = \frac{9}{20}x^{-1/4}y^{1/4},$$

$$\lambda = \frac{15}{200}x^{3/4}y^{-3/4} = \frac{3}{40}x^{3/4}y^{-3/4}.$$

Therefore, we must have

$$\frac{9}{20}x^{-1/4}y^{1/4} = \frac{3}{40}x^{3/4}y^{-3/4}.$$

To solve for y in terms of x, let us multiply both sides of this equation by $x^{1/4}y^{3/4}$:

$$\frac{9}{20}y = \frac{3}{40}x$$

or

$$y = \frac{1}{6}x.$$

Inserting this result in (L-3), we find that

$$100x + 200\left(\frac{1}{6}x\right) = 30{,}000$$
$$\frac{400x}{3} = 30{,}000$$
$$x = 225.$$

Hence,

$$y = \frac{225}{6} = 37.5.$$

So maximum production is achieved by the use of 225 units of labor and 37.5 units of capital.

▶ **Now Try Exercise 15**

In Example 3, it turns out that, at the optimum values of x and y,

$$\lambda = \frac{9}{20}x^{-1/4}y^{1/4} = \frac{9}{20}(225)^{-1/4}(37.5)^{1/4} \approx .2875,$$

$$\frac{\partial f}{\partial x} = 45x^{-1/4}y^{1/4} = 45(225)^{-1/4}(37.5)^{1/4}, \tag{6}$$

$$\frac{\partial f}{\partial y} = 15x^{3/4}y^{-3/4} = 15(225)^{3/4}(37.5)^{-3/4}. \tag{7}$$

It can be shown that the Lagrange multiplier λ can be interpreted as the *marginal productivity of money.* That is, if 1 additional dollar is available, approximately .2875 additional units of the product can be produced.

Recall that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are called the *marginal productivity of labor* and *capital*, respectively. From equations (6) and (7) we have

$$\frac{[\text{marginal productivity of labor}]}{[\text{marginal productivity of capital}]} = \frac{45(225)^{-1/4}(37.5)^{1/4}}{15(225)^{3/4}(37.5)^{-3/4}}$$
$$= \frac{45}{15}(225)^{-1}(37.5)^{1}$$
$$= \frac{3(37.5)}{225} = \frac{37.5}{75} = \frac{1}{2}.$$

On the other hand,

$$\frac{[\text{cost per unit of labor}]}{[\text{cost per unit of capital}]} = \frac{100}{200} = \frac{1}{2}.$$

This result illustrates the following law of economics. *If labor and capital are at their optimal levels, the ratio of their marginal productivities equals the ratio of their unit costs.*

The method of Lagrange multipliers generalizes to functions of any number of variables. For instance, we can maximize $f(x, y, z)$, subject to the constraint equation $g(x, y, z) = 0$, by considering the Lagrange function

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z).$$

The analogs of equations (L-1) to (L-3) are

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial \lambda} = 0.$$

Let us now show how we can solve the heat-loss problem of Section 3 by using this method.

EXAMPLE 4 **Lagrange Multipliers in Three Variables** Use Lagrange multipliers to find the values of x, y, z that minimize the objective function

$$f(x, y, z) = 11xy + 14yz + 15xz,$$

subject to the constraint

$$xyz = 147{,}840.$$

SOLUTION The Lagrange function is

$$F(x, y, z, \lambda) = 11xy + 14yz + 15xz + \lambda(147{,}840 - xyz).$$

The conditions for a relative minimum are

$$\frac{\partial F}{\partial x} = 11y + 15z - \lambda yz = 0,$$

$$\frac{\partial F}{\partial y} = 11x + 14z - \lambda xz = 0,$$

$$\frac{\partial F}{\partial z} = 14y + 15x - \lambda xy = 0,$$

$$\frac{\partial F}{\partial \lambda} = 147{,}840 - xyz = 0. \tag{8}$$

From the first three equations, we have

$$\left.\begin{aligned} \lambda &= \frac{11y + 15z}{yz} = \frac{11}{z} + \frac{15}{y} \\ \lambda &= \frac{11x + 14z}{xz} = \frac{11}{z} + \frac{14}{x} \\ \lambda &= \frac{14y + 15x}{xy} = \frac{14}{x} + \frac{15}{y} \end{aligned}\right\}. \tag{9}$$

Let us equate the first two expressions for λ:

$$\frac{11}{z} + \frac{15}{y} = \frac{11}{z} + \frac{14}{x}$$

$$\frac{15}{y} = \frac{14}{x}$$

$$x = \frac{14}{15}y.$$

Next, we equate the second and third expressions for λ in (9):

$$\frac{11}{z} + \frac{14}{x} = \frac{14}{x} + \frac{15}{y}$$

$$\frac{11}{z} = \frac{15}{y}$$

$$z = \frac{11}{15}y.$$

We now substitute the expressions for x and z in the constraint equation (8) and obtain

$$\frac{14}{15}y \cdot y \cdot \frac{11}{15}y = 147{,}840$$

$$y^3 = \frac{(147{,}840)(15)^2}{(14)(11)} = 216{,}000$$

$$y = 60.$$

From this, we find that

$$x = \frac{14}{15}(60) = 56 \quad \text{and} \quad z = \frac{11}{15}(60) = 44.$$

We conclude that the heat loss is minimized when $x = 56$, $y = 60$, and $z = 44$.

▶ Now Try Exercise 17

In the solution of Example 4, we found that, at the optimal values of x, y, and z,

$$\frac{14}{x} = \frac{15}{y} = \frac{11}{z}.$$

Referring to Example 2 of Section 1, we see that 14 is the combined heat loss through the east and west sides of the building, 15 is the heat loss through the north and south sides of the building, and 11 is the heat loss through the floor and roof. Thus, we have that, under optimal conditions,

$$\frac{[\text{heat loss through east and west sides}]}{[\text{distance between east and west sides}]} = \frac{[\text{heat loss through north and south sides}]}{[\text{distance between north and south sides}]} = \frac{[\text{heat loss through floor and roof}]}{[\text{distance between floor and roof}]}.$$

This is a principle of optimal design: Minimal heat loss occurs when the distance between each pair of opposite sides is some fixed constant times the heat loss from the pair of sides.

The value of λ in Example 4 corresponding to the optimal values of x, y, and z is

$$\lambda = \frac{11}{z} + \frac{15}{y} = \frac{11}{44} + \frac{15}{60} = \frac{1}{2}.$$

We can show that the Lagrange multiplier λ is the marginal heat loss with respect to volume. That is, if a building of volume slightly more than 147,840 cubic feet is optimally designed, $\frac{1}{2}$ unit of additional heat will be lost for each additional cubic foot of volume.

Check Your Understanding 4

1. Let $F(x, y, \lambda) = 2x + 3y + \lambda(90 - 6x^{1/3}y^{2/3})$. Find $\dfrac{\partial F}{\partial x}$.

2. Refer to Exercise 29 of Section 3. What is the function $F(x, y, l, \lambda)$ when the exercise is solved by means of the method of Lagrange multipliers?

EXERCISES 4

Solve the following exercises by the method of Lagrange multipliers.

1. Minimize $x^2 + 3y^2 + 10$, subject to the constraint $8 - x - y = 0$.

2. Maximize $x^2 - y^2$, subject to the constraint $2x + y - 3 = 0$.

3. Maximize $x^2 + xy - 3y^2$, subject to the constraint $2 - x - 2y = 0$.

4. Minimize $\frac{1}{2}x^2 - 3xy + y^2 + \frac{1}{2}$, subject to the constraint $3x - y - 1 = 0$.

5. Find the values of x, y that maximize
$$-2x^2 - 2xy - \tfrac{3}{2}y^2 + x + 2y,$$
subject to the constraint $x + y - \frac{5}{2} = 0$.

6. Find the values of x, y that minimize
$$x^2 + xy + y^2 - 2x - 5y,$$
subject to the constraint $1 - x + y = 0$.

7. **Maximizing a Product** Find the two positive numbers whose product is 25 and whose sum is as small as possible.

8. **Maximizing Area** Four hundred eighty dollars are available to fence in a rectangular garden. The fencing for the north and south sides of the garden costs \$10 per foot and the fencing for the east and west sides costs \$15 per foot. Find the dimensions of the largest possible garden.

9. **Maximizing Volume** Three hundred square inches of material are available to construct an open rectangular box with a square base. Find the dimensions of the box that maximize the volume.

10. **Minimizing Space in a Firm** The amount of space required by a particular firm is $f(x, y) = 1000\sqrt{6x^2 + y^2}$, where x and y are, respectively, the number of units of labor and capital utilized. Suppose that labor costs \$480 per unit and capital costs \$40 per unit and that the firm has \$5000 to spend. Determine the amounts of labor and capital that should be utilized in order to minimize the amount of space required.

11. **Inscribed Rectangle with Maximum Area** Find the dimensions of the rectangle of maximum area that can be inscribed in the unit circle. [See Fig. 2(a).]

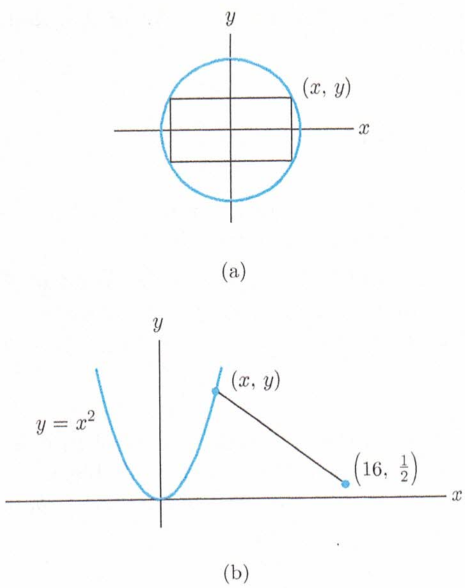


Figure 2

12. **Distance from a Point to a Parabola** Find the point on the parabola $y = x^2$ that has minimal distance from the point $(16, \frac{1}{2})$. [See Fig. 2(b).] [*Suggestion:* If d denotes the distance from (x, y) to $(16, \frac{1}{2})$, then $d^2 = (x-16)^2 + (y-\frac{1}{2})^2$. If d^2 is minimized, then d will be minimized.]

13. **Production Schedule and Production Possibilities Curve** Suppose that a firm makes two products, A and B, that use the same raw materials. Given a fixed amount of raw materials and a fixed amount of labor, the firm must decide how much of its resources should be allocated to the production of A and how much to B. If x units of A and y units of B are produced, suppose that x and y must satisfy

$$9x^2 + 4y^2 = 18{,}000.$$

The graph of this equation (for $x \geq 0$, $y \geq 0$) is called a *production possibilities curve* (Fig. 3). A point (x, y) on this curve represents a *production schedule* for the firm, committing it to produce x units of A and y units of B. The reason for the relationship between x and y involves the limitations on personnel and raw materials available to the firm. Suppose that each unit of A yields a \$3 profit, whereas each unit of B yields a \$4 profit. Then, the profit of the firm is

$$P(x, y) = 3x + 4y.$$

Find the production schedule that maximizes the profit function $P(x, y)$.

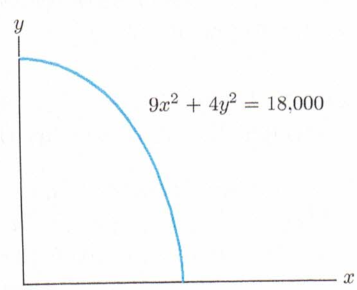


Figure 3 A production possibilities curve.

14. **Maximizing Profit** A firm makes x units of product A and y units of product B and has a production possibilities curve given by the equation $4x^2 + 25y^2 = 50{,}000$ for $x \geq 0$, $y \geq 0$. (See Exercise 13.) Suppose profits are \$2 per unit for product A and \$10 per unit for product B. Find the production schedule that maximizes the total profit.

15. **Optimal Amount of Labor** The production function for a firm is $f(x, y) = 64x^{3/4}y^{1/4}$, where x and y are the number of units of labor and capital utilized. Suppose that labor costs \$96 per unit and capital costs \$162 per unit and that the firm decides to produce 3456 units of goods.
    - **(a)** Determine the amounts of labor and capital that should be utilized in order to minimize the cost. That is, find the values of x, y that minimize $96x + 162y$, subject to the constraint $3456 - 64x^{3/4}y^{1/4} = 0$.
    - **(b)** Find the value of λ at the optimal level of production.
    - **(c)** Show that, at the optimal level of production, we have

$$\frac{\text{[marginal productivity of labor]}}{\text{[marginal productivity of capital]}} = \frac{\text{[unit price of labor]}}{\text{[unit price of capital]}}.$$

16. **Maximizing Profit** Consider the firm of Example 2, Section 3, who sells its goods in two countries. Suppose that the firm must set the same price in each country. That is, $97 - (x/10) = 83 - (y/20)$. Find the values of x and y that maximize profits under this new restriction.

17. **Maximizing a Product** Find the values of x, y, and z that maximize xyz subject to the constraint $36 - x - 6y - 3z = 0$.

18. Find the values of x, y, and z that maximize $xy + 3xz + 3yz$ subject to the constraint $9 - xyz = 0$.

19. Find the values of x, y, z that maximize

$$3x + 5y + z - x^2 - y^2 - z^2,$$

subject to the constraint $6 - x - y - z = 0$.

20. Find the values of x, y, z that minimize

$$x^2 + y^2 + z^2 - 3x - 5y - z,$$

subject to the constraint $20 - 2x - y - z = 0$.

21. **Minimizing Cost** The material for a closed rectangular box costs \$2 per square foot for the top and \$1 per square foot for the sides and bottom. Using Lagrange multipliers, find the dimensions for which the volume of the box is 12 cubic feet and the cost of the materials is minimized. [Refer to Fig. 4(a); the cost will be $3xy + 2xz + 2yz$.]

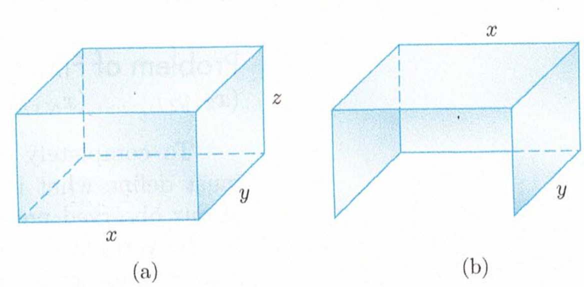


Figure 4

22. Use Lagrange multipliers to find the three positive numbers whose sum is 15 and whose product is as large as possible.

23. **Minimizing Surface Area** Find the dimensions of an open rectangular glass tank of volume 32 cubic feet for which the amount of material needed to construct the tank is minimized. [See Fig. 4(a).]

24. **Maximizing Volume** A shelter for use at the beach has a back, two sides, and a top made of canvas. [See Fig. 4(b).] Find the dimensions that maximize the volume and require 96 square feet of canvas.

25. **Production Function** Let $f(x, y)$ be any production function where x represents labor (costing \$$a$ per unit) and y represents capital (costing \$$b$ per unit). Assuming that \$$c$ is available, show that, at the values of x, y that maximize production,

$$\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = \frac{a}{b}.$$

Note: Let $F(x, y, \lambda) = f(x, y) + \lambda(c - ax - by)$. The result follows from (L-1) and (L-2).

26. **Production Function** By applying the result in Exercise 25 to the production function $f(x, y) = kx^{\alpha}y^{\beta}$, show that, for the values of x, y that maximize production, we have

$$\frac{y}{x} = \frac{a\beta}{b\alpha}.$$

(This tells us that the ratio of capital to labor does not depend on the amount of money available, nor on the level of production, but only on the numbers a, b, α, and β.)

Solutions to Check Your Understanding 4

1. The function can be written as

$$F(x, y, \lambda) = 2x + 3y + \lambda \cdot 90 - \lambda \cdot 6x^{1/3}y^{2/3}.$$

When differentiating with respect to x, treat both y and λ as constants (so $\lambda \cdot 90$ and $\lambda \cdot 6$ are also regarded as constants).

$$\frac{\partial F}{\partial x} = 2 - \lambda \cdot 6 \cdot \frac{1}{3}x^{-2/3} \cdot y^{2/3}$$
$$= 2 - 2\lambda x^{-2/3}y^{2/3}$$

(*Note:* It is not necessary to write out the multiplication by λ as we did. Most people just do this mentally and then differentiate.)

2. The quantity to be maximized is the volume xyl. The constraint is that length plus girth is 84. This translates to $84 = l + 2x + 2y$ or $84 - l - 2x - 2y = 0$. Therefore,

$$F(x, y, l, \lambda) = xyl + \lambda(84 - l - 2x - 2y).$$

5 The Method of Least Squares

Today, people can compile graphs of literally thousands of different quantities: the purchasing value of the dollar as a function of time, the pressure of a fixed volume of air as a function of temperature, the average income of people as a function of their years of formal education, or the incidence of strokes as a function of blood pressure. The observed points on such graphs tend to be irregularly distributed due to the complicated nature of the phenomena underlying them, as well as to errors made in observation. (For example, a given procedure for measuring average income may not count certain groups.)

In spite of the imperfect nature of the data, we are often faced with the problem of making assessments and predictions based on them. Roughly speaking, this problem amounts to filtering the sources of errors in the data and isolating the basic underlying trend. Frequently, on the basis of a suspicion or a working hypothesis, we may suspect that the underlying trend is linear; that is, the data should lie on a straight line. But which straight line? This is the problem that the *method of least squares* attempts to answer. To be more specific, let us consider the following problem:

Problem of Fitting a Straight Line to Data Given observed data points (x_1, y_1), (x_2, y_2), …, (x_N, y_N) on a graph, find the straight line that best fits these points.

To completely understand the statement of the problem being considered, we must define what it means for a line to "best" fit a set of points. If (x_i, y_i) is one of our observed points, we will measure how far it is from a given line $y = Ax + B$ by the vertical distance from the point to the line. Since the point on the line with x-coordinate x_i is $(x_i, Ax_i + B)$, this vertical distance is the distance between the y-coordinates $Ax_i + B$ and y_i. (See Fig. 1.) If $E_i = (Ax_i + B) - y_i$, either E_i or $-E_i$

is the vertical distance from (x_i, y_i) to the line. To avoid this ambiguity, we work with the square of this vertical distance:

$$E_i^2 = (Ax_i + B - y_i)^2.$$

The total error in approximating the data points $(x_1, y_1), \ldots, (x_N, y_N)$ by the line $y = Ax + B$ is usually measured by the sum E of the squares of the vertical distances from the points to the line,

$$E = E_1^2 + E_2^2 + \cdots + E_N^2.$$

E is called the *least-squares error* of the observed points with respect to the line. If all the observed points lie on the line $y = Ax + B$, all E_i are zero and the error E is zero. If a given observed point is far away from the line, the corresponding E_i^2 is large and hence makes a large contribution to the error E.

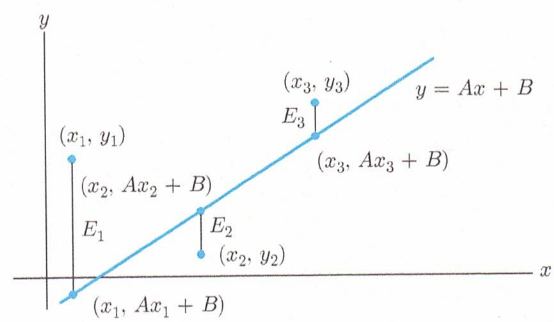


Figure 1 Fitting a line to data points.

In general, we cannot expect to find a line $y = Ax + B$ that fits the observed points so well that the error E is zero. Actually, this situation will occur only if the observed points lie on a straight line. However, we can rephrase our original problem as follows:

Problem Given observed data points $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$, find a straight line $y = Ax + B$ for which the error E is as small as possible. This line is called the *least-squares line* or *regression line.*

It turns out that this problem is a minimization problem in the two variables A and B and can be solved with the methods of Section 3. Let us consider an example.

EXAMPLE 1 **Least Squares Error** Find the straight line that minimizes the least-squares error for the points $(1, 4)$, $(2, 5)$, $(3, 8)$.

SOLUTION Let the straight line be $y = Ax + B$. When $x = 1, 2, 3$, the y-coordinate of the corresponding point of the line is $A+B$, $2A+B$, $3A+B$, respectively. Therefore, the squares of the vertical distances from the points $(1, 4)$, $(2, 5)$, $(3, 8)$ are, respectively,

$$E_1^2 = (A + B - 4)^2,$$
$$E_2^2 = (2A + B - 5)^2,$$
$$E_3^2 = (3A + B - 8)^2.$$

(See Fig. 2.) Thus, the least-squares error is

$$E = E_1^2 + E_2^2 + E_3^2 = (A + B - 4)^2 + (2A + B - 5)^2 + (3A + B - 8)^2.$$

This error obviously depends on the choice of A and B. Let $f(A, B)$ denote this least-squares error. We want to find values of A and B that minimize $f(A, B)$. To do so, we take partial derivatives with respect to A and B and set the partial derivatives equal to zero:

$$\frac{\partial f}{\partial A} = 2(A+B-4) + 2(2A+B-5)\cdot 2 + 2(3A+B-8)\cdot 3$$

$$= 28A + 12B - 76 = 0,$$

$$\frac{\partial f}{\partial B} = 2(A+B-4) + 2(2A+B-5) + 2(3A+B-8)$$

$$= 12A + 6B - 34 = 0.$$

To find A and B, we must solve the system of simultaneous linear equations

$$28A + 12B = 76$$

$$12A + \ 6B = 34.$$

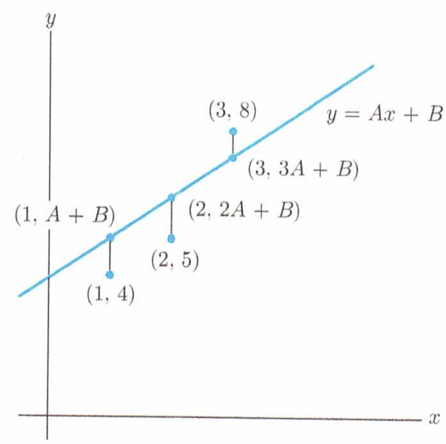


Figure 2

Multiplying the second equation by 2 and subtracting from the first equation, we have $4A = 8$, or $A = 2$. Therefore, $B = \frac{5}{3}$, and the straight line that minimizes the least-squares error is $y = 2x + \frac{5}{3}$.

▶ *Now Try Exercise 1*

The minimization process used in Example 1 can be applied to a general set of data points $(x_1, y_1), \ldots, (x_N, y_N)$ to obtain the following algebraic formula for A and B:

$$A = \frac{N\cdot\Sigma\, xy - \Sigma\, x\cdot\Sigma\, y}{N\cdot\Sigma\, x^2 - (\Sigma\, x)^2},$$

$$B = \frac{\Sigma\, y - A\cdot\Sigma\, x}{N},$$

where

$\Sigma\, x$ = sum of the x-coordinates of the data points

$\Sigma\, y$ = sum of the y-coordinates of the data points

$\Sigma\, xy$ = sum of the products of the coordinates of the data points

$\Sigma\, x^2$ = sum of the squares of the x-coordinates of the data points

N = number of data points.

That is,

$$\Sigma\, x = x_1 + x_2 + \cdots + x_N$$

$$\Sigma\, y = y_1 + y_2 + \cdots + y_N$$

$$\Sigma\, xy = x_1\cdot y_1 + x_2\cdot y_2 + \cdots + x_N\cdot y_N$$

$$\Sigma\, x^2 = x_1^2 + x_2^2 + \cdots + x_N^2.$$

EXAMPLE 2 Car-Accident-Related Deaths in the U.S. The following table gives the number in thousands of car-accident-related deaths in the U.S. for certain years.

Year	Number (in thousands)
1990	46.8
2000	43.4
2005	45.3
2007	43.9
2008	39.7
2009	35.9

(a) Use the preceding formulas to obtain the straight line that best fits these data.

(b) Use the straight line found in part (a) to estimate the number of car-accident-related deaths in 2012. (It is interesting that, while the number of drivers is obviously increasing with time, the number of car-accident-related deaths is actually decreasing, maybe because of improvements in car-manufacturing technologies and added safety measures.)

SOLUTION (a) The data are plotted in Fig. 3, where x denotes the number of years since 1990. The sums are calculated in Table 1 and then used to determine the values of A and B.

TABLE 1 Car-Accident-Related Deaths in U.S.

x Years since 1990	y Number of deaths in thousands	xy	x^2
0	46.8	0	0
10	43.4	434	100
15	45.3	679.5	225
17	43.9	746.3	289
18	39.7	714.6	324
19	35.9	682.1	361
$\sum x = 79$	$\sum y = 255$	$\sum xy = 3256.5$	$\sum x^2 = 1299$

$$A = \frac{N \cdot 3256.5 - 79 \cdot 255}{6 \cdot 1299 - 79^2} \approx -.39$$

$$B = \frac{255 + .39 \cdot 79}{6} \approx 47.64$$

Therefore, the equation of the least-squares line is $y = -.39x + 47.64$ (see Fig. 3).

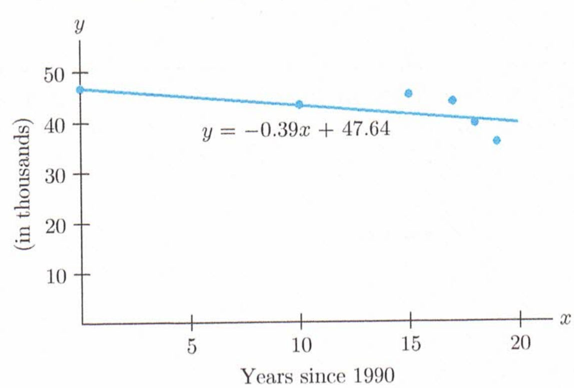


Figure 3

(b) We use the straight line to estimate the number of car-accident-related deaths in 2012 by setting $x = 22$. Then, we get

$$y = (-.39)(22) + 47.64 = 39.06.$$

Therefore, we estimate the number of car-related accidental deaths to be 39.06 thousand in 2012.

▶ *Now Try Exercise 9*

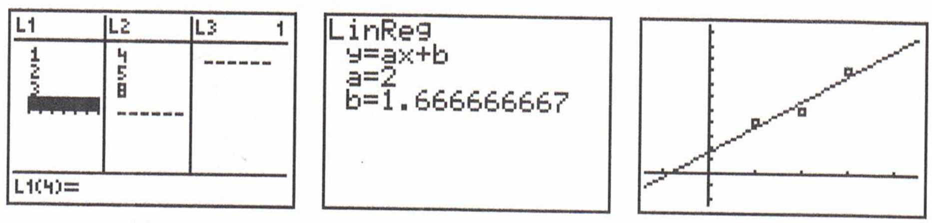
INCORPORATING TECHNOLOGY

Least-Squares Method To implement the least-squares method on your TI-83/84, select [STAT] [1] for the EDIT screen to obtain a table used for entering the data. If necessary, clear data from columns L_1 and/or L_2 by moving the cursor to the top of the column and pressing [CLEAR] [ENTER]. [See Fig. 4(a).]

After the x- and y-values are placed in lists on a graphing calculator, we use the statistical routine LinReg to calculate the coefficients of the least-squares line. Now press [STAT] [▷] for the CALC menu, and press [4] to place **LinReg(ax+b)** on the home screen. Press [ENTER] to obtain the slope and y-intercept of the least-squares line. [See Fig. 4(b).]

If desired, we can automatically assign the equation for the line to a function and graph it along with the original points. First, we assign the equation for the least-squares line to a function. Select [Y=], move to the function, and press [CLEAR] to erase any current expression. Now, press [VARS] [5] to select the **Statistics** variables. Move your cursor over to the EQ menu, and press [1] for **RegEQ** (Regression Equation).

To graph this line, press [GRAPH]. To graph this line along with the original data points, we proceed as follows. From the [Y=], and with only the least-squares line selected, press [2nd] [STAT PLOT] [ENTER] to select **Plot1**, and press [ENTER] to turn **Plot1 ON**. Now, select the first plot from the six icons for the plot **Type**. This corresponds to a scatter plot. Finally, press [GRAPH]. [See Fig. 4(c).]



(a) (b) (c)

Figure 4

Check Your Understanding 5

1. Let $E = (A + B + 2)^2 + (3A + B)^2 + (6A + B - 8)^2$. What is $\dfrac{\partial E}{\partial A}$?

2. Find the formula (of the type in Problem 1) that gives the least-squares error E for the points $(1, 10)$, $(5, 8)$, and $(7, 0)$.

EXERCISES 5

1. Find the least-squares error E for the least-squares line fit to the four points in Fig. 5.

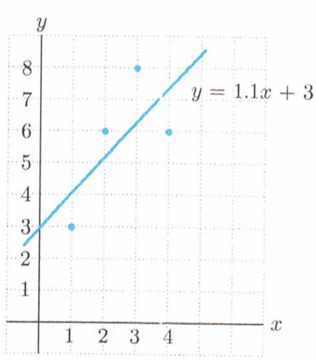


Figure 5

2. Find the least-squares error E for the least-squares line fit to the five points in Fig. 6.

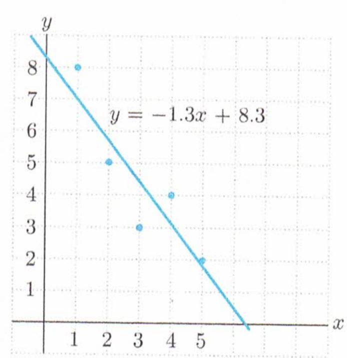


Figure 6

3. Find the formula (of the type in Check Your Understanding Problem 1) that gives the least-squares error for the points $(2, 6)$, $(5, 10)$, and $(9, 15)$.

4. Find the formula (of the type in Check Your Understanding Problem 1) that gives the least-squares error for the points $(8, 4)$, $(9, 2)$, and $(10, 3)$.

In Exercises 5–8, use partial derivatives to obtain the formula for the best least-squares fit to the data points.

5. $(1, 2)$, $(2, 5)$, $(3, 11)$

6. $(1, 8)$, $(2, 4)$, $(4, 3)$

7. $(1, 9), (2, 8), (3, 6), (4, 3)$

8. $(1, 5), (2, 7), (3, 6), (4, 10)$

9. Complete Table 2 and find the values of A and B for the straight line that provides the best least-squares fit to the data.

TABLE 2

x	y	xy	x^2
1	7		
2	6		
3	4		
4	3		
$\Sigma x =$	$\Sigma y =$	$\Sigma xy =$	$\Sigma x^2 =$

10. Complete Table 3 and find the values of A and B for the straight line that provides the best least-squares fit to the data.

TABLE 3

x	y	xy	x^2
1	2		
2	3		
3	7		
4	9		
5	12		
$\Sigma x =$	$\Sigma y =$	$\Sigma xy =$	$\Sigma x^2 =$

In the remaining exercises, use one or more of the three methods discussed in this section (partial derivatives, formulas, or graphing utilities) to obtain the formula for the least-squares line.

11. **Health Care Expenditures** Table 4 gives the U.S. per capita health-care expenditures for the years 2005–2009. (*Source: Health Care Financing Review.*)

TABLE 4 U.S. Per Capita Health Care Expenditures

Years (after 2000)	Dollars
5	6,259
6	7,073
7	7,437
8	7,720
9	7,960

(a) Find the least-squares line for these data.

(b) Use the least-squares line to predict the per capita health care expenditures for the year 2012.

(c) Use the least-squares line to predict when per capita health care expenditures will reach $10,000.

12. Table 5 shows the 2012 price of a gallon of fuel (in U.S. dollars) and the average miles driven per automobile for several countries. (*Source: International Energy Annual and Highway Statistics.*)

TABLE 5 Effect of Gas Prices on Miles Driven

Country	Price per Gallon	Average Miles per Auto
Canada	$4.42	10,000
England	$8.15	8,430
Germany	$7.37	7,700
United States	$3.71	15,000

(a) Find the straight line that provides the best least-squares fit to these data.

(b) In 2012, the price of gas in France was $5.54 per gallon. Use the straight line of part (a) to estimate the average number of miles automobiles were driven in France.

13. Table 6 gives the U.S. minimum wage in dollars for certain years.

TABLE 6 U.S. Federal Minimum Wage

Year	1985	1990	1995	2000	2005	2010
Wage	3.35	3.80	4.25	5.15	5.15	7.25

(a) Use the method of least squares to obtain the straight line that best fits these data. [*Hint:* First convert *Year* to *Years after 1980.*]

(b) Estimate the minimum wage for the year 1998.

(c) If the trend determined by the straight line in part (a) continues, when will the minimum wage reach $10?

14. Table 7 gives the number of cars (in millions) in use in the United States for certain years. (*Source: Motor Vehicle Facts and Figures.*)

TABLE 7 Automobile Population

Year	Cars	Year	Cars
1990	193.1	2006	250.8
1995	205.4	2007	254.4
2000	225.8	2008	255.9
2005	247.4	2009	254.2

(a) Use the method of least squares to obtain the straight line that best fits these data. [*Hint:* First convert *Years* to *Years after 1990.*]

(b) Estimate the number of cars in use in 1997.

(c) If the trend determined by the straight line in part (a) continues, when will the number of cars in use reach 275 million?

15. An ecologist wished to know whether certain species of aquatic insects have their ecological range limited by temperature. He collected the data in Table 8, relating the average daily temperature at different portions of a creek with the elevation (above sea level) of that portion of the creek. (The authors express their thanks to Dr. J. David Allen, formerly of the Department of Zoology at the University of Maryland, for providing the data for this exercise.)

(a) Find the straight line that provides the best least-squares fit to these data.

(b) Use the linear function to estimate the average daily temperature for this creek at altitude 3.2 kilometers.

TABLE 8 Relationship between Elevation and Temperature in a Creek

Elevation (kilometers)	Average Temperature (degrees Celsius)
2.7	11.2
2.8	10
3.0	8.5
3.5	7.5

Solutions to Check Your Understanding 5

1.
$$\begin{aligned}\frac{\partial E}{\partial A} &= 2(A+B+2)\cdot 1 + 2(3A+B)\cdot 3 \\ &\quad + 2(6A+B-8)\cdot 6 \\ &= (2A+2B+4) + (18A+6B) \\ &\quad + (72A+12B-96) \\ &= 92A+20B-92\end{aligned}$$

(Notice that we used the general power rule when differentiating and so had to always multiply by the derivative of the quantity inside the parentheses. Also, you might be tempted to first square the terms in the expression for E and then differentiate. We recommend that you resist this temptation.)

2. $E = (A+B-10)^2 + (5A+B-8)^2 + (7A+B)^2$. In general, E is a sum of squares, one for each point being fitted. The point (a, b) gives rise to the term $(aA+B-b)^2$.

6 Double Integrals

Up to this point, our discussion of the calculus of several variables has been confined to the study of differentiation. Let us now take up the topic of the integration of functions of several variables. As has been the case throughout most of this chapter, we restrict our discussion to functions $f(x, y)$ of two variables.

We begin with some motivation. Before we define the concept of an integral for functions of several variables, we review the essential features of the integral in one variable.

Consider the definite integral $\int_a^b f(x)\,dx$. To write down this integral takes two pieces of information. The first is the function $f(x)$. The second is the interval over which the integration is to be performed. In this case, the interval is the portion of the x-axis from $x = a$ to $x = b$. The value of the definite integral is a number. In case the function $f(x)$ is nonnegative throughout the interval from $x = a$ to $x = b$, this number equals the area under the graph of $f(x)$ from $x = a$ to $x = b$. (See Fig. 1.) If $f(x)$ is negative for some values of x in the interval, the integral still equals the area bounded by the graph, but areas below the x-axis are counted as negative.

Let us generalize the foregoing ingredients to a function $f(x, y)$ of two variables. First, we must provide a two-dimensional analog of the interval from $x = a$ to $x = b$. This is easy. We take a two-dimensional region R of the plane, such as the region shown in Fig. 2. As our generalization of $f(x)$, we take a function $f(x, y)$ of two variables. Our generalization of the definite integral is denoted

$$\iint_R f(x, y)\,dx\,dy$$

and is called the *double integral of $f(x, y)$ over the region R*. The value of the double integral is a number defined as follows. For the sake of simplicity, let us begin by assuming that $f(x, y) \geq 0$ for all points (x, y) in the region R. [This is the analog of the assumption that $f(x) \geq 0$ for all x in the interval from $x = a$ to $x = b$.] This

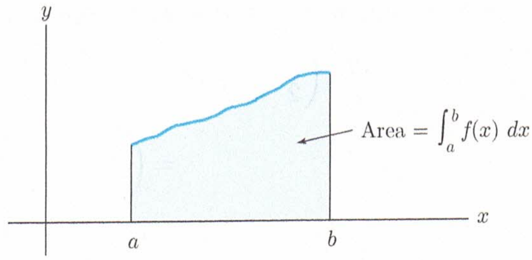


Figure 1

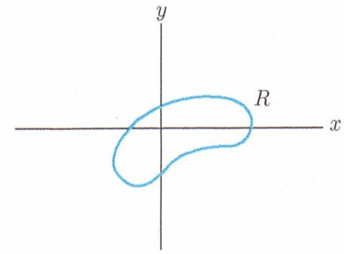


Figure 2 A region in the xy-plane.

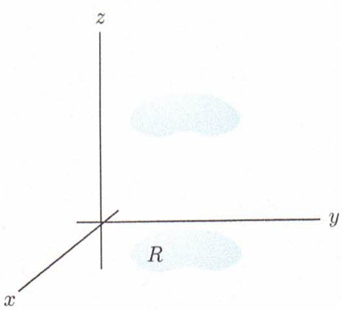


Figure 3 Graph of $f(x, y)$ above the region R.

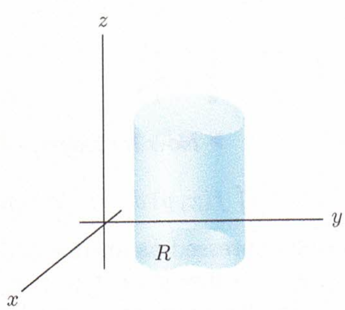


Figure 4 Solid bounded by $f(x, y)$ over R.

means that the graph of f lies above the region R in three-dimensional space. (See Fig. 3.) The portion of the graph over R determines a solid figure. (See Fig. 4.) This figure is called the *solid bounded by $f(x, y)$ over the region R*. We define the double integral $\iint\limits_R f(x, y)\,dx\,dy$ to be the volume of this solid. In case the graph of $f(x, y)$ lies partially above the region R and partially below, we define the double integral to be the volume of the solid above the region minus the volume of the solid below the region. That is, we count volumes below the xy-plane as negative.

Now that we have defined the notion of a double integral, we must learn how to calculate its value. To do so, let us introduce the notion of an iterated integral. Let $f(x, y)$ be a function of two variables, let $g(x)$ and $h(x)$ be two functions of x alone, and let a and b be numbers. Then, an *iterated integral* is an expression of the form

$$\int_a^b \left(\int_{g(x)}^{h(x)} f(x, y)\,dy \right) dx.$$

To explain the meaning of this collection of symbols, we proceed from the inside out. We evaluate the integral

$$\int_{g(x)}^{h(x)} f(x, y)\,dy$$

by considering $f(x, y)$ as a function of y alone. This is indicated by the dy in the inner integral. We treat x as a constant in this integration. So, we evaluate the integral by first finding an antiderivative $F(x, y)$ with respect to y. The integral above is then evaluated as

$$F(x, h(x)) - F(x, g(x)).$$

That is, we evaluate the antiderivative between the limits $y = g(x)$ and $y = h(x)$. This gives us a function of x alone. To complete the evaluation of the integral, we integrate this function from $x = a$ to $x = b$. The next two examples illustrate the procedure for evaluating iterated integrals.

EXAMPLE 1 **A Double Integral** Evaluate the iterated integral

$$\int_1^2 \left(\int_3^4 (y - x)\,dy \right) dx.$$

SOLUTION Here $g(x)$ and $h(x)$ are constant functions: $g(x) = 3$ and $h(x) = 4$. We evaluate the inner integral first. The variable in this integral is y, so we treat x as a constant.

$$\begin{aligned}\int_3^4 (y - x)\,dy &= \left(\frac{1}{2}y^2 - xy\right)\Big|_3^4 \\ &= \left(\frac{1}{2}\cdot 16 - x\cdot 4\right) - \left(\frac{1}{2}\cdot 9 - x\cdot 3\right) \\ &= 8 - 4x - \frac{9}{2} + 3x \\ &= \frac{7}{2} - x\end{aligned}$$

Now, we carry out the integration with respect to x:

$$\int_1^2 \left(\frac{7}{2} - x\right) dx = \frac{7}{2}x - \frac{1}{2}x^2 \Big|_1^2$$
$$= \left(\frac{7}{2}\cdot 2 - \frac{1}{2}\cdot 4\right) - \left(\frac{7}{2} - \frac{1}{2}\cdot 1\right)$$
$$= (7 - 2) - (3) = 2.$$

So, the value of the iterated integral is 2. ▶ **Now Try Exercise 1**

EXAMPLE 2 **A Double Integral** Evaluate the iterated integral

$$\int_0^1 \left(\int_{\sqrt{x}}^{x+1} 2xy\, dy\right) dx.$$

SOLUTION We evaluate the inner integral first.

$$\int_{\sqrt{x}}^{x+1} 2xy\, dy = xy^2 \Big|_{\sqrt{x}}^{x+1} = x(x+1)^2 - x(\sqrt{x})^2$$
$$= x(x^2 + 2x + 1) - x \cdot x$$
$$= x^3 + 2x^2 + x - x^2$$
$$= x^3 + x^2 + x$$

Now, we evaluate the outer integral.

$$\int_0^1 (x^3 + x^2 + x)\, dx = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{13}{12}$$

So, the value of the iterated integral is $\frac{13}{12}$. ▶ **Now Try Exercise 7**

Let us now return to the discussion of the double integral $\iint\limits_R f(x, y)\, dx\, dy$. When the region R has a special form, the double integral may be expressed as an iterated integral, as follows: Suppose that R is bounded by the graphs of $y = g(x)$, $y = h(x)$ and by the vertical lines $x = a$ and $x = b$. (See Fig. 5.) In this case, we have the following fundamental result, which we cite without proof.

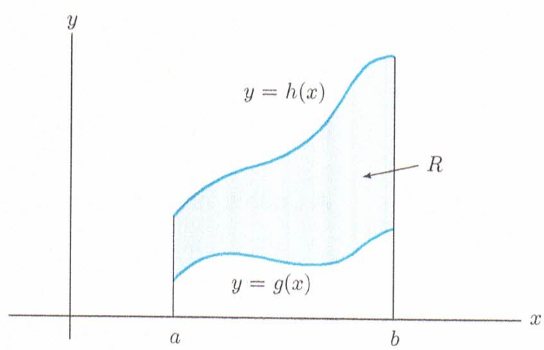


Figure 5

Let R be the region in the xy-plane bounded by the graphs of $y = g(x)$, $y = h(x)$, and the vertical lines $x = a$, $x = b$. Then,

$$\iint\limits_R f(x, y)\, dx\, dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x, y)\, dy\right) dx.$$

Since the value of the double integral gives the volume of the solid bounded by the graph of $f(x, y)$ over the region R, the preceding result may be used to calculate volumes, as the next two examples show.

EXAMPLE 3 **Volume Using a Double Integral** Calculate the volume of the solid bounded above by the function $f(x, y) = y - x$ and lying over the rectangular region R: $1 \le x \le 2$, $3 \le y \le 4$. (See Fig. 6.)

SOLUTION The desired volume is given by the double integral $\iint\limits_R (y - x)\, dx\, dy$. By the result just cited, this double integral is equal to the iterated integral

$$\int_1^2 \left(\int_3^4 (y - x)\, dy \right) dx.$$

The value of this iterated integral was shown in Example 1 to be 2, so the volume of the solid shown in Fig. 6 is 2. ▶ **Now Try Exercise 13**

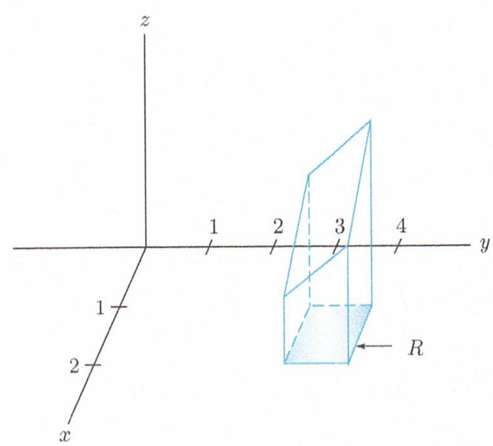


Figure 6

EXAMPLE 4 **Double Integral over a Region** Calculate $\iint\limits_R 2xy\, dx\, dy$, where R is the region shown in Fig. 7.

SOLUTION The region R is bounded below by $y = \sqrt{x}$, above by $y = x + 1$, on the left by $x = 0$, and on the right by $x = 1$. Therefore,

$$\iint\limits_R 2xy\, dx\, dy = \int_0^1 \left(\int_{\sqrt{x}}^{x+1} 2xy\, dy \right) dx = \frac{13}{12} \quad \text{(by Example 2).}$$

▶ **Now Try Exercise 9**

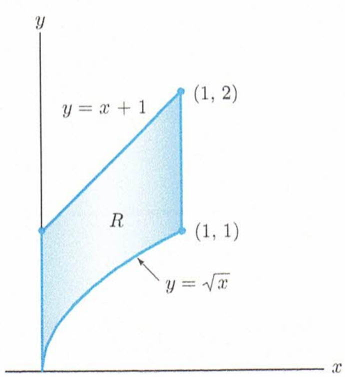


Figure 7

In our discussion, we have confined ourselves to iterated integrals in which the inner integral was with respect to y. In a completely analogous manner, we may treat iterated integrals in which the inner integral is with respect to x. Such iterated integrals may be used to evaluate double integrals over regions R bounded by curves of the form $x = g(y)$, $x = h(y)$ and horizontal lines $y = a$, $y = b$. The computations are analogous to those given in this section.

Check Your Understanding 6

1. Calculate the iterated integral

$$\int_0^2 \left(\int_0^{x/2} e^{2y-x}\, dy \right) dx.$$

2. Calculate $\iint\limits_R e^{2y-x}\, dx\, dy$, where R is the region in Fig. 8.

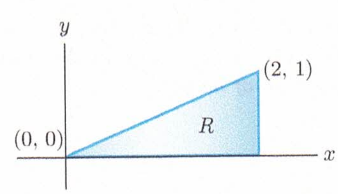


Figure 8

EXERCISES 6

Calculate the following iterated integrals.

1. $\int_0^1 \left(\int_0^1 e^{x+y}\, dy \right) dx$
2. $\int_{-1}^1 \left(\int_{-1}^1 xy\, dx \right) dy$
3. $\int_{-2}^0 \left(\int_{-1}^1 xe^{xy}\, dy \right) dx$
4. $\int_0^1 \left(\int_{-1}^1 \frac{1}{3}y^3 x\, dy \right) dx$
5. $\int_1^4 \left(\int_x^{x^2} xy\, dy \right) dx$
6. $\int_0^3 \left(\int_x^{2x} y\, dy \right) dx$
7. $\int_{-1}^1 \left(\int_x^{2x} (x+y)\, dy \right) dx$
8. $\int_0^1 \left(\int_0^x e^{x+y}\, dy \right) dx$

Let R be the rectangle consisting of all points (x, y) such that $0 \le x \le 2$, $2 \le y \le 3$. Calculate the following double integrals. Interpret each as a volume.

9. $\iint_R xy^2\, dx\, dy$
10. $\iint_R (xy + y^2)\, dx\, dy$
11. $\iint_R e^{-x-y}\, dx\, dy$
12. $\iint_R e^{y-x}\, dx\, dy$

Calculate the volumes over the following regions R bounded above by the graph of $f(x, y) = x^2 + y^2$.

13. R is the rectangle bounded by the lines $x = 1$, $x = 3$, $y = 0$, and $y = 1$.
14. R is the region bounded by the lines $x = 0$, $x = 1$ and the curves $y = 0$ and $y = \sqrt[3]{x}$.

Solutions to Check Your Understanding 6

1. $$\int_0^2 \left(\int_0^{x/2} e^{2y-x}\, dy \right) dx = \int_0^2 \left(\frac{1}{2}e^{2y-x} \Big|_0^{x/2} \right) dx$$
$$= \int_0^2 \left(\frac{1}{2}e^{2(x/2)-x} - \frac{1}{2}e^{2(0)-x} \right) dx$$
$$= \int_0^2 \left(\frac{1}{2} - \frac{1}{2}e^{-x} \right) dx$$
$$= \frac{1}{2}x + \frac{1}{2}e^{-x} \Big|_0^2$$
$$= \frac{1}{2}\cdot 2 + \frac{1}{2}e^{-2} - \left(\frac{1}{2}\cdot 0 + \frac{1}{2}e^{-0} \right)$$
$$= 1 + \frac{1}{2}e^{-2} - 0 - \frac{1}{2}$$
$$= \frac{1}{2} + \frac{1}{2}e^{-2}$$

2. The line passing through the points $(0, 0)$ and $(2, 1)$ has equation $y = x/2$. Hence, the region R is bounded below by $y = 0$, above by $y = x/2$, on the left by $x = 0$, and on the right by $x = 2$. Therefore,
$$\iint_R e^{2y-x}\, dx\, dy = \int_0^2 \left(\int_0^{x/2} e^{2y-x}\, dy \right) dx$$
$$= \frac{1}{2} + \frac{1}{2}e^{-2}$$
by Problem 1.

▸ Summary

KEY TERMS AND CONCEPTS	EXAMPLES

1 Examples of Functions of Several Variables

A function $f(x, y)$ of the two variables x and y is a rule that assigns a number to each pair of values for the variables.

Given a function of two variables $f(x, y)$, the graph of the equation $f(x, y) = c$ is a curve in the xy-plane called the *level curve of height c*.

Suppose that, during a certain time period, the number of units of goods produced with x units of labor and y units of capital is given by the Cobb–Douglas production function $f(x, y) = 40x^{1/2}y^{1/2}$.

(a) How many units of goods will be produced with 16 units of labor and 16 units of capital?

(b) Determine the isoquant or level curve at level 100 of the production function.

KEY TERMS AND CONCEPTS	EXAMPLES

Solution

(a) $f(16,16) = 40(16)^{1/2} \cdot (16)^{1/2} = 40 \cdot 4 \cdot 4 = 640$; there will be 640 units of goods produced.

(b) The level curve is the graph of $f(x,y) = 100$, or

$$40x^{1/2}y^{1/2} = 100$$
$$y^{1/2} = \frac{100}{40x^{1/2}} = \frac{5}{2x^{1/2}}$$
$$y = \frac{25}{4x}.$$

Thus, the isoquant is the hyperbola $y = \frac{25}{4x}$. Each point on the hyperbola has coordinates $(x, \frac{25}{4x})$ and represents a combination of capital x and labor $\frac{25}{4x}$ that yields 100 units of production.

2 Partial Derivatives

The *partial derivative of $f(x,y)$ with respect to x*, written $\frac{\partial f}{\partial x}$, is the derivative of $f(x,y)$, where y is treated as a constant and $f(x,y)$ is considered as a function of x alone. The *partial derivative of $f(x,y)$ with respect to y*, written $\frac{\partial f}{\partial y}$, is the derivative of $f(x,y)$, where x is treated as a constant. Higher-order derivatives are defined similarly.

Let $f(x,y) = e^{x^2+7y}$.

(a) Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial x}(1,-1)$.

(b) Compute $\frac{\partial^2 f}{\partial x^2}$.

Solution

(a) Thinking of y as a constant, we have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{x^2+7y}\right] = e^{x^2+7y} \cdot \frac{\partial}{\partial x}\left[x^2+7y\right] = 2x\, e^{x^2+7y}.$$

Thinking of x as a constant, we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{x^2+7y}\right] = e^{x^2+7y} \cdot \frac{\partial}{\partial y}\left[x^2+7y\right] = 7e^{x^2+7y}.$$

Finally,

$$\frac{\partial f}{\partial x}(1,-1) = 2x\, e^{x^2+7y}\Big|_{(1,-1)} = 2e^{1-7} = 2e^{-6}.$$

(b) Start with the formula for $\frac{\partial f}{\partial x}$, and think of y as a constant; then,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}[2x\, e^{x^2+7y}]\\ &= e^{x^2+7y}(2) + 2x\frac{\partial}{\partial x}[e^{x^2+7y}] \quad \text{(Product Rule)}\\ &= 2e^{x^2+7y} + 2xe^{x^2+7y}\frac{\partial}{\partial x}(x^2+7y)\\ &= 2e^{x^2+7y} + 2xe^{x^2+7y}(2x)\\ &= 2e^{x^2+7y}(1+2x^2).\end{aligned}$$

3 Maxima and Minima of Functions of Several Variables

First-Derivative Test for Functions of Two Variables If $f(x,y)$ has either a relative maximum or minimum at $(x,y) = (a,b)$, then

$$\frac{\partial f}{\partial x}(a,b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a,b) = 0.$$

You can use this test to locate candidate points where the function has a relative extreme value. Once you have located a candidate point where the first derivatives are 0, you check whether this point is a

Let $f(x, y) = x^2 + y^2 - 4x - 6y + 10$.

(a) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(b) Find the points (x, y) where the first derivatives are zero.

(c) Apply the second derivative test at the points in (b) and decide, if possible, the nature of $f(x, y)$ at each of these points.

KEY TERMS AND CONCEPTS	EXAMPLES

maximum, a minimum, or neither by applying the second-derivative test for function of two variables. The outcomes of the second-derivative test depend on the signs of $\frac{\partial^2 f}{\partial x^2}(a, b)$ and

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \, \partial y}\right)^2.$$

(See Section 3 for the full statement and the examples for an illustration.)

Solution

(a) $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^2 + y^2 - 4x - 6y + 10\right] = 2x - 4$$

$$\frac{\partial f}{\partial y} = 2y - 6$$

(b) The partial derivatives are equal to 0 when $x = 2$ and $y = 3$, so the only point where both partial derivatives are 0 is (2, 3).

(c) The nature of the function at the point (2, 3) depends on the signs of $D(2, 3)$ and $\frac{\partial^2 f}{\partial x^2}(2, 3)$. We have

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(2y - 6) = 0.$$

So,

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \, \partial y}\right)^2 = (2)(2) - 0 = 4.$$

Since $D(2, 3) = 4 > 0$ and $\frac{\partial^2 f}{\partial x^2}(2, 3) = 2 > 0$, according to the second-derivative test, $f(x, y)$ has a relative minimum at (2, 3).

4 Lagrange Multipliers and Constrained Optimization

To find a relative maximum or minimum of the function $f(x, y)$ subject to the constraint $g(x, y) = 0$, we can use the Lagrange multiplier method. We apply this method in steps as follows:

Step 1 Form the function

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

The number λ is called a *Lagrange multiplier.*

Step 2 Compute the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$.

Step 3 Solve the system of equations

$$\frac{\partial F}{\partial x} = 0; \frac{\partial F}{\partial y} = 0; g(x, y) = 0.$$

Step 4 If you found more than one point (x, y) in Step 3, evaluate f at all the points. The largest of these values is the maximum value for f, and the smallest is the minimum value of f.

Use Lagrange multipliers to find the minimum value of $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 4$.

Solution

Step 1 Write the constraint in the form $x + y - 4 = 0$. Then, $g(x, y) = x + y - 4$ and $F(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 4)$.

Step 2 $$\frac{\partial F}{\partial x} = 2x + \lambda; \frac{\partial F}{\partial y} = 2y + \lambda$$

Step 3 Solve

$$\begin{cases} 2x + \lambda = 0 & (1) \\ 2y + \lambda = 0 & (2) \\ x + y - 4 = 0 & (3) \end{cases}$$

Subtract (2) from (1) and get $2x - 2y = 0$ or $x = y$. Use $x = y$ in (3) and get $2y = 4$ or $y = 2$. Hence $x = 2$.

Step 4 At the point (2, 2), f takes on the value 8, which is the minimum value of $x^2 + y^2$ subject to $x + y = 4$

5 The Method of Least Squares

Suppose that you have a set of N data points $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$, and you have reasons to believe that y is linearly related to x, or at least approximately. Then, you can look for a linear function $y = Ax + B$ that best fits the given data. This line is called the *least-squares line* or *regression line.* The coefficients A and B are computed as follows:

$$A = \frac{N \cdot \Sigma xy - \Sigma x \cdot \Sigma y}{N \cdot \Sigma x^2 - (\Sigma x)^2}$$

$$B = \frac{\Sigma y - A \cdot \Sigma x}{N},$$

Table 1 shows the number of seniors who graduated from a high school in Jefferson City, Missouri, in the years 2007 to 2012.

(a) Find the line that best fits these data.

(b) Use the straight line that you found in part (a) to approximate the number of seniors who will graduate in 2013.

KEY TERMS AND CONCEPTS

where

Σx = sum of the x-coordinates of the data points

Σy = sum of the y-coordinates of the data points

Σxy = sum of the products of the coordinates of the data points

Σx^2 = sum of the squares of the x-coordinates of the data points

N = number of data points.

EXAMPLES

TABLE 1 Students' Data

Years (after 2000)	Graduating seniors
7	245
8	275
9	225
10	215
11	218
12	212

Solution

(a) Let x denote the number of years since 2000 and y the number of graduating seniors. Then, our data points are

$$(7, 245), (8, 275), (9, 225), (10, 215), (11, 218), (12, 212).$$

Let $y = Ax + B$ denote the line of best fit through these points. The sums are calculated in Table 2 and then used to determine the values of A and B.

TABLE 2 Students' Data

x	y	xy	x^2
7	245	1715	49
8	275	2200	64
9	225	2025	81
10	215	2150	100
11	218	2398	121
12	212	2544	144
$\sum x = 57$	$\sum y = 1390$	$\sum xy = 13032$	$\sum x^2 = 559$

In this example, $N = 6$, since we have six data points. Applying the formulas for the coefficients, we find

$$\begin{aligned} A &= \frac{N \cdot \Sigma xy - \Sigma x \cdot \Sigma y}{N \cdot \Sigma x^2 - (\Sigma x)^2} \\ &= \frac{6 \cdot 13{,}032 - 57 \cdot 1390}{6 \cdot 559 - (57)^2} \\ &= -\frac{346}{35} \approx -9.89 \\ B &= \frac{\Sigma y - A \cdot \Sigma x}{N} \\ &= \frac{1390 + \frac{346}{35} \cdot 57}{6} \\ &= \frac{34{,}186}{105} \approx 325.58. \end{aligned}$$

Thus, the line of best fit is $y = -9.89x + 325.58$, which is a decreasing line.

(b) To approximate the number of graduating seniors in 2013, we set $x = 13$ in the formula for the line of best fit and get

$$y = -9.89(13) + 325.58 \approx 197.01$$

Thus, in 2013, the number of graduating seniors will be approximately 197.

KEY TERMS AND CONCEPTS	EXAMPLES

6 Double Integrals

For a function of two variables $f(x, y)$, we can define a double integral where (x, y) varies over a region R in the xy-plane. In the iterated double integral

$$\int_{-2}^{2}\int_{0}^{3} f(x, y)\,dx\,dy,$$

the inner variable x varies from 0 to 3, while the outer variable y varies from -2 to 2. The point (x, y) in this double integral varies over the rectangular region $0 \le x \le 3,\ -2 \le y \le 2$. Evaluate the integral.

If, in the iterated integral, the symbol $dy\,dx$ appears, instead of $dx\,dy$, then you should integrate with respect to y first. This is the case with the iterated integral

$$\int_{-3}^{2}\int_{1}^{5} f(x, y)\,dy\,dx.$$

The inner integral, which you should compute first, is with respect to y.

Evaluate the iterated integral $\int_0^1\int_0^2 xy^2\,dx\,dy$.

Solution

Step 1 Evaluate the inner integral in x, while treating y as a constant.

$$\int_0^2 xy^2\,dx = \frac{y^2}{2}x^2\Big|_0^2 = \frac{y^2}{2}(2^2 - 0) = 2y^2$$

Step 2 Evaluate the outer integral of the function $2y^2$ that we found in Step 1.

$$\int_0^1 2y^2\,dy = \frac{2}{3}y^3\Big|_0^1 = \frac{2}{3}$$

Thus, the double integral is equal to $\frac{2}{3}$.

Fundamental Concept Check Exercises

1. Give an example of a level curve of a function of two variables.
2. Explain how to find a first partial derivative of a function of two variables.
3. Explain how to find a second partial derivative of a function of two variables.
4. What expression involving a partial derivative gives an approximation to $f(a+h,b) - f(a,b)$?
5. Interpret $\frac{\partial f}{\partial y}(2,3)$ as a rate of change.
6. Give an example of a Cobb–Douglas production function. What is the marginal productivity of labor? Of capital?
7. Explain how to find possible relative extreme points for a function of several variables.
8. State the second-derivative test for functions of two variables.
9. Outline how the method of Lagrange multipliers is used to solve an optimization problem.
10. What is the least-squares line approximation to a set of data points? How is the line determined?
11. Give a geometric interpretation for $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) \ge 0$.
12. Give a formula for evaluating a double integral in terms of an iterated integral.

Review Exercises

1. Let $f(x,y) = x\sqrt{y}/(1+x)$. Compute $f(2,9)$, $f(5,1)$, and $f(0,0)$.
2. Let $f(x,y,z) = x^2 e^{y/z}$. Compute $f(-1,0,1)$, $f(1,3,3)$, and $f(5,-2,2)$.
3. **A Savings Account** If A dollars are deposited in a bank at a 6% continuous interest rate, the amount in the account after t years is $f(A,t) = Ae^{.06t}$. Find and interpret $f(10, 11.5)$.
4. Let $f(x,y,\lambda) = xy + \lambda(5 - x - y)$. Find $f(1,2,3)$.
5. Let $f(x,y) = 3x^2 + xy + 5y^2$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
6. Let $f(x,y) = 3x - \frac{1}{2}y^4 + 1$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
7. Let $f(x,y) = e^{x/y}$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
8. Let $f(x,y) = x/(x-2y)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
9. Let $f(x,y,z) = x^3 - yz^2$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.
10. Let $f(x,y,\lambda) = xy + \lambda(5 - x - y)$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial \lambda}$.

11. Let $f(x,y) = x^3y + 8$. Compute $\frac{\partial f}{\partial x}(1,2)$ and $\frac{\partial f}{\partial y}(1,2)$.

12. Let $f(x,y,z) = (x+y)z$. Evaluate $\frac{\partial f}{\partial y}$ at $(x,y,z) = (2,3,4)$.

13. Let $f(x,y) = x^5 - 2x^3y + \frac{1}{2}y^4$. Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x\,\partial y}$, and $\frac{\partial^2 f}{\partial y\,\partial x}$.

14. Let $f(x,y) = 2x^3 + x^2y - y^2$. Compute $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x\,\partial y}$ at $(x,y) = (1,2)$.

15. A dealer in a certain brand of electronic calculator finds that (within certain limits) the number of calculators she can sell per week is given by $f(p,t) = -p + 6t - .02pt$, where p is the price of the calculator and t is the number of dollars spent on advertising. Compute $\frac{\partial f}{\partial p}(25, 10{,}000)$ and $\frac{\partial f}{\partial t}(25, 10{,}000)$, and interpret these numbers.

16. The crime rate in a certain city can be approximated by a function $f(x,y,z)$, where x is the unemployment rate, y is the number of social services available, and z is the size of the police force. Explain why $\frac{\partial f}{\partial x} > 0$, $\frac{\partial f}{\partial y} < 0$, and $\frac{\partial f}{\partial z} < 0$.

In Exercises 17–20, find all points (x,y) where $f(x,y)$ has a possible relative maximum or minimum.

17. $f(x,y) = -x^2 + 2y^2 + 6x - 8y + 5$

18. $f(x,y) = x^2 + 3xy - y^2 - x - 8y + 4$

19. $f(x,y) = x^3 + 3x^2 + 3y^2 - 6y + 7$

20. $f(x,y) = \frac{1}{2}x^2 + 4xy + y^3 + 8y^2 + 3x + 2$

In Exercises 21–23, find all points (x,y) where $f(x,y)$ has a possible relative maximum or minimum. Then, use the second-derivative test to determine, if possible, the nature of $f(x,y)$ at each of these points. If the second-derivative test is inconclusive, so state.

21. $f(x,y) = x^2 + 3xy + 4y^2 - 13x - 30y + 12$

22. $f(x,y) = 7x^2 - 5xy + y^2 + x - y + 6$

23. $f(x,y) = x^3 + y^2 - 3x - 8y + 12$

24. Find the values of x, y, z at which

$$f(x,y,z) = x^2 + 4y^2 + 5z^2 - 6x + 8y + 3$$

assumes its minimum value.

Use the method of Lagrange multipliers to:

25. Maximize $3x^2 + 2xy - y^2$, subject to the constraint $5 - 2x - y = 0$.

26. Find the values of x, y that minimize $-x^2 - 3xy - \frac{1}{2}y^2 + y + 10$, subject to the constraint $10 - x - y = 0$.

27. Find the values of x, y, z that minimize $3x^2 + 2y^2 + z^2 + 4x + y + 3z$, subject to the constraint $4 - x - y - z = 0$.

28. Find the dimensions of a rectangular box of volume 1000 cubic inches for which the sum of the dimensions is minimized.

29. A person wants to plant a rectangular garden along one side of a house and put a fence on the other three sides. (See Fig. 1.) Using the method of Lagrange multipliers, find the dimensions of the garden of greatest area that can be enclosed with 40 feet of fencing.

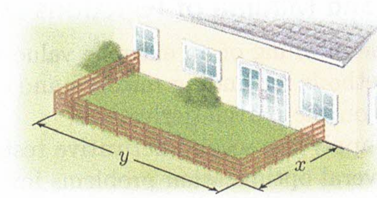


Figure 1 A garden.

30. The solution to Exercise 29 is $x = 10$, $y = 20$, $\lambda = 10$. If 1 additional foot of fencing becomes available, compute the new optimal dimensions and the new area. Show that the increase in area (compared with the area in Exercise 29) is approximately equal to 10 (the value of λ).

In Exercises 31–33, find the straight line that best fits the following data points, where "best" is meant in the sense of least squares.

31. $(1,1)$, $(2,3)$, $(3,6)$

32. $(1,1)$, $(3,4)$, $(5,7)$

33. $(0,1)$, $(1,-1)$, $(2,-3)$, $(3,-5)$

In Exercises 34 and 35, calculate the iterated integral.

34. $\int_0^1 \left(\int_0^4 (x\sqrt{y} + y)\,dy \right) dx$

35. $\int_0^5 \left(\int_1^4 (2xy^4 + 3)\,dy \right) dx$

In Exercises 36 and 37, let R be the rectangle consisting of all points (x,y), such that $0 \le x \le 4$, $1 \le y \le 3$, and calculate the double integral.

36. $\iint_R (2x + 3y)\,dx\,dy$

37. $\iint_R 5\,dx\,dy$

38. The present value of y dollars after x years at 15% continuous interest is $f(x,y) = ye^{-.15x}$. Sketch some sample level curves. (Economists call this collection of level curves a *discount system*.)

Learning Objectives

1 Examples of Functions of Several Variables

- Introduce functions of two and three variables.
- Discuss examples of functions of several variables.
- Evaluate functions of several variables.
- Discuss applications in architectural design and economics.
- Define level curves of functions of several variables and discuss some of their applications.

2 Partial Derivatives

- Define a partial derivative of a function of several variables.
- Compute and evaluate partial derivatives.
- Interpret a partial derivative as a rate of change.
- Approximate a function using partial derivatives.
- Discuss applications of partial derivatives.

3 Maxima and Minima of Functions of Several Variables

- Explain the meaning of an extreme value for a function of several variables.
- Discuss methods for finding maxima and minima of functions of several variables based on conditions on the partial derivatives.
- State and apply the second-derivative test for finding extreme values of functions of two variables.
- Present several optimization problems involving extreme values of functions of several variables.

4 Lagrange Multipliers and Constrained Optimization

- Explain optimization problems with constraints involving functions of several variables.
- Show how to solve optimization problems with constraints using the method of Lagrange multipliers.
- Use Lagrange multipliers to solve applied optimization problems.

5 The Method of Least Squares

- Introduce the technique of fitting a straight line through a given set of data.
- Explain how to measure the least-squares error when fitting a curve through data.
- Discuss the least-squares line, or regression line, that minimizes the least-squares error.
- Explain how partial derivatives of functions of several variables can be used to find the regression line.
- Discuss applications of least-squares method to analyzing and predicting data.

6 Double Integrals

- Present the concept of double integrals of functions of two variables.
- Evaluate examples of double integrals using iterated integrals.
- Discuss regions in the plane and explain their role in evaluating double integrals.
- Compute volumes of solids using double integrals.

Sources

Section 2

1. Exercise 32, from Stone, R. (1945). The analysis of market demand. *Journal of the Royal Statistical Society, 108*, 286–391.
2. Exercise 38, from Routh, J. (1971). *Mathematical preparation for laboratory technicians.* Philadelphia, PA: W. B. Saunders, p. 92.

Section 3

1. Example 3, from March, L. (1972). Elementary models of built forms. In L. Martin & L. March (Eds.), *Urban space and structures.* New York, NY: Cambridge University Press.

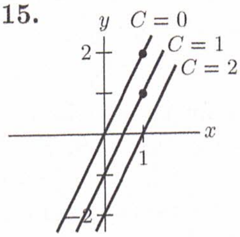
Section 5

1. Exercise 11, from U.S. Health Care Financing Administration, *Health Care Financing Review*, Spring 2012.

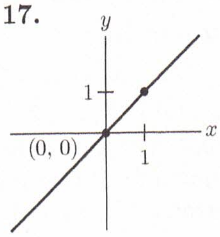
Answers

Exercises 1

1. $f(5,0)=25$, $f(5,-2)=51$, $f(a,b)=a^2-3ab-b^2$ **3.** $g(2,3,4)=-2$, $g(7,46,44)=7/2$ **5.** $f(2+h,3)=3h+6, f(2,3)=6, f(2+h,3)-f(2,3)=3h$ **7.** $C(x,y,z)=6xy+10xz+10yz$ **9.** $f(8,1)=40$, $f(1,27)=180$, $f(8,27)=360$ **11.** $\approx$\$50. \$50 invested at 5% continuously compounded interest will yield \$100 in 13.8 years **13. (a)** \$1875 **(b)** \$2250; yes **15.**



17.



19. $f(x,y)=y-3x$ **21.** They correspond to the points having the same altitude above sea level. **23.** (d) **25.** (c)

Exercises 2

1. $5y$, $5x$ **3.** $4xe^y$, $2x^2e^y$ **5.** $\frac{1}{y}-\frac{y}{x^2}$; $\frac{-x}{y^2}+\frac{1}{x}$ **7.** $4(2x-y+5)$, $-2(2x-y+5)$ **9.** $(2xe^{3x}+3x^2e^{3x})\ln y$, x^2e^{3x}/y **11.** $\frac{2y}{(x+y)^2}$, $-\frac{2x}{(x+y)^2}$ **13.** $\frac{3}{2}\sqrt{\frac{K}{L}}$ **15.** $\frac{2xy}{z}$, $\frac{x^2}{z}$, $-\frac{1+x^2y}{z^2}$ **17.** ze^{yz}, xz^2e^{yz}, $x(yz+1)e^{yz}$ **19.** 1, 3 **21.** $\frac{\partial f}{\partial y}=2xy$, $\frac{\partial f}{\partial y}(2,-1)=-4$. If x is kept constant at 2 and y is allowed to vary near -1, then $f(x,y)$ changes at a rate -4 times the change in y. **23.** $\frac{\partial f}{\partial x}=3x^2y+2y^2$, $\frac{\partial^2 f}{\partial x^2}=6xy$, $\frac{\partial f}{\partial y}=x^3+4xy$, $\frac{\partial^2 f}{\partial y^2}=4x$, $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}=3x^2+4y$ **25. (a)** Marginal productivity of labor $=480$; of capital $=40$ **(b)** $480h$ **(c)** Production decreases by 240 units. **27.** If the price of a bus ride increases and the price of a train ticket remains constant, fewer people will ride the bus. An increase in train ticket prices coupled with constant bus fare should cause more people to ride the bus. **29.** If the average price of audio files increases and the average price of an MP3 player remains constant, people will purchase fewer audio files. An increase in average MP3 player prices coupled with constant audio files prices should cause a decline in the number of MP3 players purchased. **31.** $\frac{\partial V}{\partial P}(20,300)=-.06$, $\frac{\partial V}{\partial T}(20,300)=.004$ **33.** $\partial f/\partial r>0$, $\partial f/\partial m>0$, $\partial f/\partial p<0$ **35.** $\frac{\partial^2 f}{\partial x^2}=-\frac{45}{4}x^{-5/4}y^{1/4}$; marginal productivity of labor is decreasing.

Exercises 3

1. $(-2,1)$ **3.** $(26,11)$ **5.** $(1,-3)$, $(-1,-3)$ **7.** $(\sqrt{5},1)$, $(\sqrt{5},-1)$, $(-\sqrt{5},1)$, $(-\sqrt{5},-1)$ **9.** $(1/3,4/3)$ **11.** Relative minimum; neither relative maximum nor relative minimum. **13.** Relative maximum; neither relative maximum nor relative minimum; relative maximum. **15.** Neither relative maximum nor relative minimum. **17.** (0, 0) Relative minimum **19.** $(-1,-4)$ relative max **21.** $(0,-1)$ relative min **23.** $(-1,2)$ relative max: (1, 2) neither max nor min **25.** $(1/4,2)$ min; $(1/4,-2)$ neither max nor min **27.** $(1/2,1/6,1/2)$ **29.** 14 in. × 14 in. × 28 in. **31.** $x=120$, $y=80$

Exercises 4

1. 58 at $x=6$, $y=2$, $\lambda=12$ **3.** 13 at $x=8$, $y=-3$, $\lambda=13$ **5.** $x=1/2$, $y=2$ **7.** 5, 5 **9.** Base 10 in., height 5 in. **11.** $F(x,y,\lambda)=4xy+\lambda(1-x^2-y^2)$; $\sqrt{2}\times\sqrt{2}$ **13.** $F(x,y,\lambda)=3x+4y+\lambda(18{,}000-9x^2-4y^2)$; $x=20$, $y=60$ **15. (a)** $F(x,y,\lambda)=96x+162y+\lambda(3456-64x^{3/4}y^{1/4})$; $x=81$, $y=16$ **(b)** $\lambda=3$ **17.** $x=12$, $y=2$, $z=4$ **19.** $x=2$, $y=3$, $z=1$ **21.** $F(x,y,z,\lambda)=3xy+2xz+2yz+\lambda(12-xyz)$; $x=2$, $y=2$, $z=3$ **23.** $F(x,y,z,\lambda)=xy+2xz+2yz+\lambda(32-xyz)$; $x=y=4$, $z=2$

Exercises 5

1. $E=6.7$ **3.** $E=(2A+B-6)^2+(5A+B-10)^2+(9A+B-15)^2$ **5.** $y=4.5x-3$ **7.** $y=-2x+11.5$ **9.** $y=-1.4x+8.5$ **11. (a)** $y=404.9x+4455.5$ **(b)** 9314 **(c)** 2014 **13. (a)** $y=.14x+2.38$ **(b)** \$4.90 per hour **(c)** $x=54.5$, or the year 2035 **15. (a)** $y=-4.24x+22.01$ **(b)** $y=8.442$ degrees Celsius

Exercises 6

1. e^2-2e+1 **3.** $2-e^{-2}-e^2$ **5.** $309\frac{3}{8}$ **7.** $5/3$ **9.** $38/3$ **11.** $e^{-5}+e^{-2}-e^{-3}-e^{-4}$ **13.** $9\frac{1}{3}$

Answers to Fundamental Concept Check Exercises

1. $f(x, y) = x^2y + x$, level curve at height 10 is the graph of $f(x, y) = 10$ or $x^2y + x = 10$. Solving for y, we find $y = \frac{10-x}{x^2}$. **2.** To find the first partial derivative with respect to x of $f(x, y)$, treat y as a constant and differentiate the formula for $f(x, y)$ with respect to x. The partial derivative with respect to y is defined similarly. **3.** To find a second partial derivative of $f(x, y)$—say, the second partial derivative with respect to x, $\frac{\partial^2 f}{\partial x^2}$—treat y as a constant and differentiate the formula for $\frac{\partial f}{\partial x}$ with respect to x. **4.** $f(a+h, b) - f(a, b) \approx \frac{\partial f}{\partial x}(a, b) \cdot h$ **5.** If x is kept constant at 2 and y is allowed to vary near 3, then $f(x, y)$ changes at a rate that is $\frac{\partial f}{\partial y}(2, 3)$ times the change in y. **6.** A Cobb–Douglas production function: $f(x, y) = 600x^{3/5}y^{2/5}$. The marginal productivity of labor is $\frac{\partial f}{\partial x}$. The marginal productivity of capital is $\frac{\partial f}{\partial y}$. **7.** Look for points where all the first partial derivatives are 0. For functions of two variables, you can apply the second derivative test. **8.** See Section 3 for the statement of the second-derivative test for functions of two variables. **9.** To find a relative maximum or minimum of the function $f(x, y)$ subject to the constraint $g(x, y) = 0$, we can use the Lagrange multiplier method as follows; Form the function $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$. Compute the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$. Solve the system of equations $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $g(x, y) = 0$. If you found more than one point (x, y) in Step 3, evaluate f at all these points. The largest of these values is the maximum value for f, and the smallest is the minimum value of f. **10.** The least-squares line approximation to a set of data points is the line that best fits the data in the sense that it minimizes the least-squares error (the sum of the squares of the distances from the given data points to the line). If the line is $y = Ax + B$, then the coefficients are computed as follows

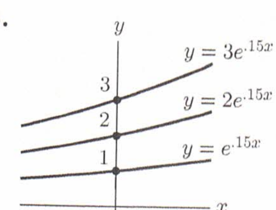
$$A = \frac{N \cdot \Sigma xy - \Sigma x \cdot \Sigma y}{N \cdot \Sigma x^2 - (\Sigma x)^2}, \qquad B = \frac{\Sigma y - A \cdot \Sigma x}{N}$$

11. $\int\int_R f(x,y)\,dx\,dy$ is the volume of the solid bounded above by $f(x,y) \geq 0$ and lying over the region R. **12.** Let R be the region in the xy-plane bounded by the graphs of $y = g(x)$, $y = h(x)$, and the vertical lines $x = a$, $x = b$. Then,

$$\int\int_R f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx.$$

Chapter Review Exercises

1. 2, 5/6, 0 **2.** $f(-1,0,1) = 1$, $f(1,3,3) = e$, $f(5,-2,2) = 25/e$ **3.** $\approx$19.94. Ten dollars increases to 20 dollars in 11.5 years. **4.** $f(1,2,3) = 8$ **5.** $6x + y, x + 10y$ **6.** $\partial f/\partial x = 3$, $\partial f/\partial y = -2y^3$ **7.** $\frac{1}{y}e^{x/y}$, $-\frac{x}{y^2}e^{x/y}$ **8.** $\frac{\partial f}{\partial x} = \frac{-2y}{(x-2y)^2}$, $\frac{\partial f}{\partial y} = \frac{2x}{(x-2y)^2}$ **9.** $3x^2, -z^2, -2yz$ **10.** $\partial f/\partial x = y - \lambda$, $\partial f/\partial y = x - \lambda$, $\partial f/\partial \lambda = 5 - x - y$ **11.** 6, 1 **12.** $\frac{\partial f}{\partial y} = z$; $\frac{\partial f}{\partial y}(2,3,4) = 4$ **13.** $20x^3 - 12xy$, $6y^2$, $-6x^2$, $-6x^2$ **14.** $\frac{\partial^2 f}{\partial x^2} = 12x + 2y$, $\frac{\partial^2 f}{\partial y^2} = -2$, $\frac{\partial^2 f}{\partial x \partial y} = 2x$, $\frac{\partial^2 f}{\partial x \partial y}(1,2) = 2$, $\frac{\partial^2 f}{\partial x^2}(1,2) = 16$ **15.** −201, 5.5. At the level $p = 25$, $t = 10{,}000$, an increase in price of \$1 will result in a loss in sales of approximately 201 calculators, and an increase in advertising of \$1 will result in the sale of approximately 5.5 additional calculators. **16.** Increases with increased unemployment and decreases with increased social services and police force size. **17.** $(3,2)$ **18.** $(2,-1)$ **19.** $(0,1)$, $(-2,1)$ **20.** $(-11,2)$, $(5,-2)$ **21.** Min at $(2,3)$ **22.** Relative minimum at $(1,3)$. **23.** Min at $(1,4)$; neither max nor min at $(-1,4)$. **24.** Minimum value at $(3,-1,0)$. **25.** 20; $x = 3$, $y = -1$ **26.** $x = 7$, $y = 3$ **27.** $x = 1/2$, $y = 3/2$, $z = 2$ **28.** $x = y = z = 10$ **29.** $F(x,y,\lambda) = xy + \lambda(40 - 2x - y)$; $x = 10$, $y = 20$ **30.** $x = 10.25$ ft, $y = 20.5$ ft **31.** $y = 5/2x - 5/3$ **32.** $y = 3/2x - 1/2$ **33.** $y = -2x + 1$ **34.** 32/3 **35.** 5160 **36.** $= 80$ **37.** 40 **38.**



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