Math 141 Spring 2025 **Midterm Exam 4** **April 4, 2025**

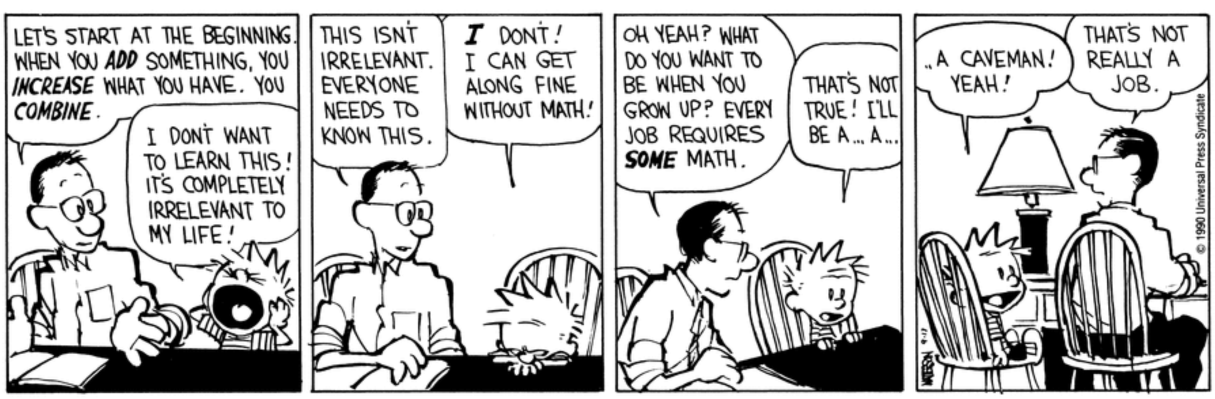
Name: Solutions

Problem	1	2 / 3	4 / 5	6 / 7	8	9	Total
Possible	8	16	16	18	22	20	100
Received							

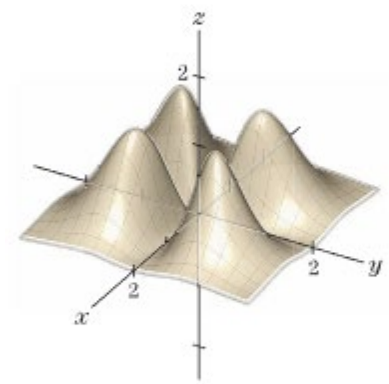
DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 × 5 card of handwritten notes and a calculator.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.

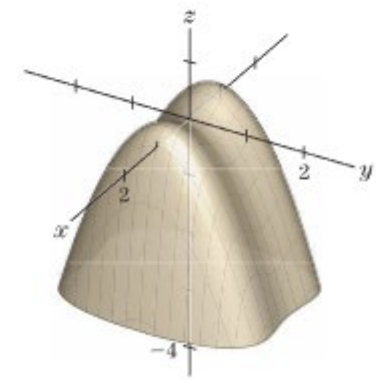


8 points 1. Match the graphs of the following four functions to the level curves show below the functions. Just write a letter (**a** or **b** or **c** or **d**) next to each.



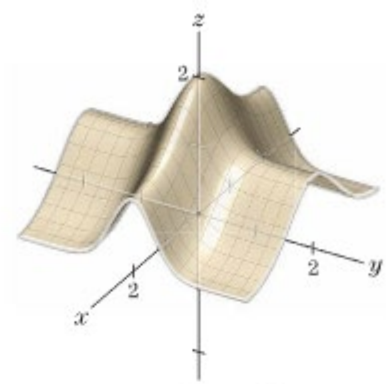
$$z = \frac{15x^2y^2e^{-x^2-y^2}}{x^2+y^2}$$

c



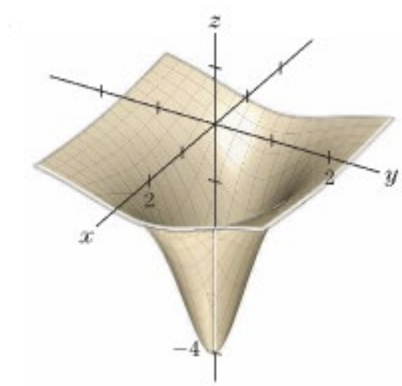
$$z = x^2 - x^4 - y^2$$

b



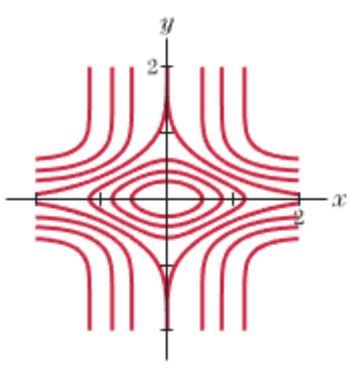
$$z = e^{-x^2} + e^{-4y^2}$$

a

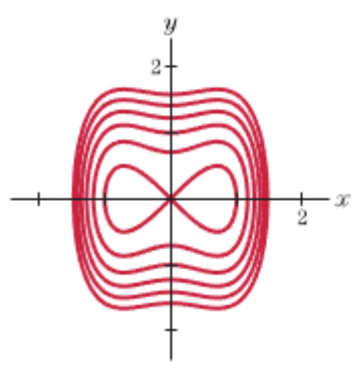


$$z = \frac{-4}{1+x^2+2y^2}$$

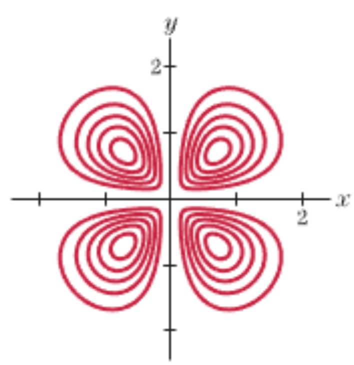
d



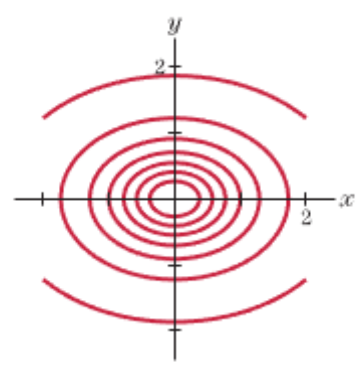
(a)



(b)



(c)



(d)

12 points 2. Consider the production function $f(x,y) = 4x^{3/4}y^{1/4}$, which gives the number of units of goods produced when x units of labor and y units of capital are used.

/4 Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$\frac{\partial f}{\partial x} = 4 \cdot \frac{3}{4} x^{-1/4} y^{1/4} = \frac{3y^{1/4}}{x^{1/4}}$

$\frac{\partial f}{\partial y} = 4x^{3/4} \cdot \frac{1}{4} y^{-3/4} = \frac{x^{3/4}}{y^{3/4}}$

/4 Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x = 16$ and $y = 81$. Note that $81^{1/4} = 3$ and $16^{1/4} = 2$.

$\frac{\partial f}{\partial x}(16,81) = \frac{3 \cdot 81^{1/4}}{16^{1/4}} = \frac{3 \cdot 3}{2} = \frac{9}{2}$

$\frac{\partial f}{\partial y}(16,81) = \frac{16^{3/4}}{81^{3/4}} = \frac{2^3}{3^3} = \frac{8}{27}$

If $y \nearrow 1$, then $f \nearrow \frac{8}{27}$ approximately

/2 Find the marginal productivity of capital of f at $x = 16$ and $y = 81$.

/2 Using above results, approximately what is $f(16,82) - f(16,81)$?

4 points 3. Suppose that $f(10,10) = 100$, $\frac{\partial f}{\partial x}(10,10) = 4$ and $\frac{\partial f}{\partial y}(10,10) = 3$. Estimate $f(12,9)$.

$f(12,9) \approx f(10,10) + 2 \cdot 4 - 1 \cdot 3$

$= 105$

4 points 4. Suppose the distance D that a car can travel depends on the amount of gas g in the car and the total weight w of the passengers in the car. Circle > 0 or $= 0$ or < 0 for the following derivatives of D.

/2 Should $\frac{\partial D}{\partial g}$ be > 0 or $= 0$ or < 0? If $g \nearrow$, then $D \nearrow$

/2 Should $\frac{\partial D}{\partial w}$ be > 0 or $= 0$ or < 0? If $w \nearrow$, then $D \searrow$

12 points 5. Find the point(s) at which $f(x, y) = -2x + 2xy - y^2 + 4y - 1$ has minimum(s) and maximum(s), and determine what type of point (min or max or neither) each point is.

$$\frac{\partial f}{\partial x} = -2 + 2y = 0 \Rightarrow y = 1$$

$$\frac{\partial f}{\partial y} = 2x - 2y + 4 = 0 \Rightarrow x = -1$$

$(-1, 1)$

$$\frac{\partial^2 f}{\partial x^2} = 0 \quad \frac{\partial^2 f}{\partial y^2} = -2 \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2$$

$$D(x, y) = (0)(-2) - 2^2 < 0$$

$\Rightarrow$ neither min nor max at $(-1, 1)$.

8 points 6. Find the following derivatives.

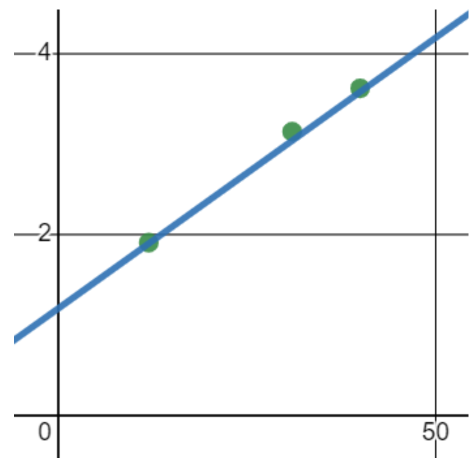
For $f(x,y) = e^{x^2 y}$

/3 $\frac{\partial f}{\partial y} = e^{x^2 y} \cdot x^2$

/5 $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(e^{x^2 y} \cdot x^2\right) = \frac{\partial}{\partial x} e^{x^2 y} \cdot x^2 + e^{x^2 y} \cdot \frac{\partial}{\partial x} x^2$

$= e^{x^2 y} \cdot 2xy \cdot x^2 + e^{x^2 y} \cdot 2x$

using the product rule.



10 points 7. Suppose that I asked three students how many hours they study per week and what their current GPA is, and found a least squares line based on their responses of

$$GPA \approx 1.2 + .1 * hours\ studied$$

/4 What do the values of 1.2 and .1 tell us?

If hrs = 0, GPA is 1.2.
For each additional hour studied,
GPA increases by .1.

/3 What GPA would result from studying 10 hours per week?

$1.2 + .1(10) = 2.2$

/3 How many hours per week would you need to study to get a GPA of 3.6?

$3.6 = 1.2 + (.1)(hrs.)$

$\Rightarrow hrs = \frac{3.6 - 1.2}{.1} = 24.$

22 points 8. Suppose we want a very tiny home with dimensions x, y and z to have volume 8 cubic yards, so $xyz = 8$. Suppose that the daily loss (through the walls, ceiling and floor) of heat is given by

$$H = 4xy + 2xz + yz = 4xy + 2x\left(\frac{8}{xy}\right) + y\left(\frac{8}{xy}\right)$$

Find the dimensions of the home which minimize heat loss H. $= 4xy + 16y^{-1} + 8x^{-1}$

For this problem, find the solution by substituting $z = \frac{8}{xy}$ into $H = 4xy + 2xz + yz$ and (1) find the values of x and y at which H is minimized, and (2) show that at these x and y values function H is indeed minimized (use second derivatives and $D(x, y)$).

$\frac{\partial H}{\partial x} = 4y + 8(-1)x^{-2} = 4y - \frac{8}{x^2} = 0 \Rightarrow y = \frac{2}{x^2}$

$\frac{\partial H}{\partial y} = 4x + 16(-1)y^{-2} = 4x - \frac{16}{y^2} = 0 \Rightarrow x = \frac{4}{y^2}$

So $x = \frac{4}{\left(\frac{2}{x^2}\right)^2} = \frac{4}{\frac{4}{x^4}} = 4 \cdot \frac{x^4}{4} = x^4$

So $x = x^4 \Rightarrow x^3 = 1 \Rightarrow x = 1 \Rightarrow y = \frac{2}{1^2} = 2$ (and $z = \frac{8}{1 \cdot 2} = 4$).

$\frac{\partial^2 H}{\partial x^2} = (-8)(-2)x^{-3} = \frac{16}{x^3}$ $\quad \frac{\partial^2 H}{\partial y^2} = (-16)(-2)y^{-3} = \frac{32}{y^3}$ $\quad \frac{\partial^2 H}{\partial x \partial y} = 4 = \frac{\partial^2 H}{\partial y \partial x}$

$D(x,y) = \frac{16}{x^3} \cdot \frac{32}{y^3} - 4^2$

$D(1,2) = \frac{16}{1^3} \cdot \frac{32}{2^3} - 16 = 16 \cdot 4 - 16 > 0$, so max or min.

$\frac{\partial^2 H}{\partial x^2}(1,2) = \frac{16}{1^3} = 16 > 0$ so min.

20 points 9. Same as previous problem: minimize

$$H = 4xy + 2xz + yz$$

with the constraint that $\underbrace{xyz = 8}$. But now solve this problem by using the Lagrange Multiplier Method. So $8 - xyz = 0$

$$F(x,y,z,\lambda) = 4xy + 2xz + yz + \lambda(8 - xyz)$$
$$= 4xy + 2xz + yz + 8\lambda - \lambda xyz$$

① $\frac{\partial F}{\partial x} = 4y + 2z - \lambda yz = 0 \Rightarrow \lambda = \frac{4y+2z}{yz} = \frac{4}{z} + \frac{2}{y}$

② $\frac{\partial F}{\partial y} = 4x + z - \lambda xz = 0 \Rightarrow \lambda = \frac{4x+z}{xz} = \frac{4}{z} + \frac{1}{x}$

③ $\frac{\partial F}{\partial z} = 2x + y - \lambda xy = 0 \Rightarrow \lambda = \frac{2x+y}{xy} = \frac{2}{y} + \frac{1}{x}$

①,②: $\frac{2}{y} = \frac{1}{x} \Rightarrow y = 2x$, i.e. $x = \frac{1}{2}y$

①,③: $\frac{4}{z} = \frac{1}{x} \Rightarrow z = 4x$, i.e. $x = \frac{1}{4}z$

In constraint:

$x \cdot 2x \cdot 4x = 8 \Rightarrow 8x^3 = 8 \Rightarrow x = 1$

$\Rightarrow y = 2 \cdot 1 = 2$

$\Rightarrow z = 4 \cdot 1 = 4$

(as in previous problem)