Math 141 Spring 2024 **Midterm Exam 2** **February 20, 2024**

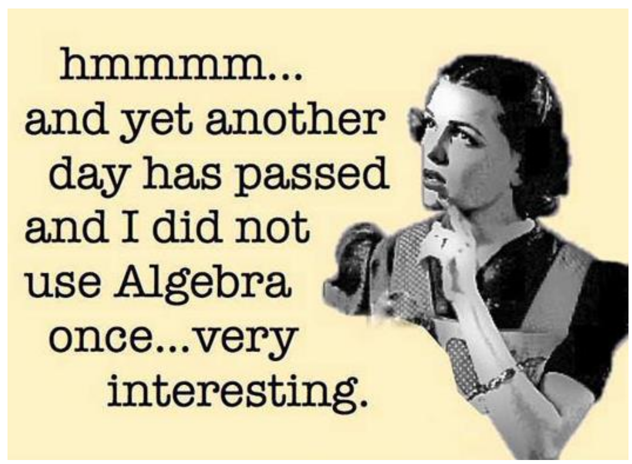
Name: Solutions

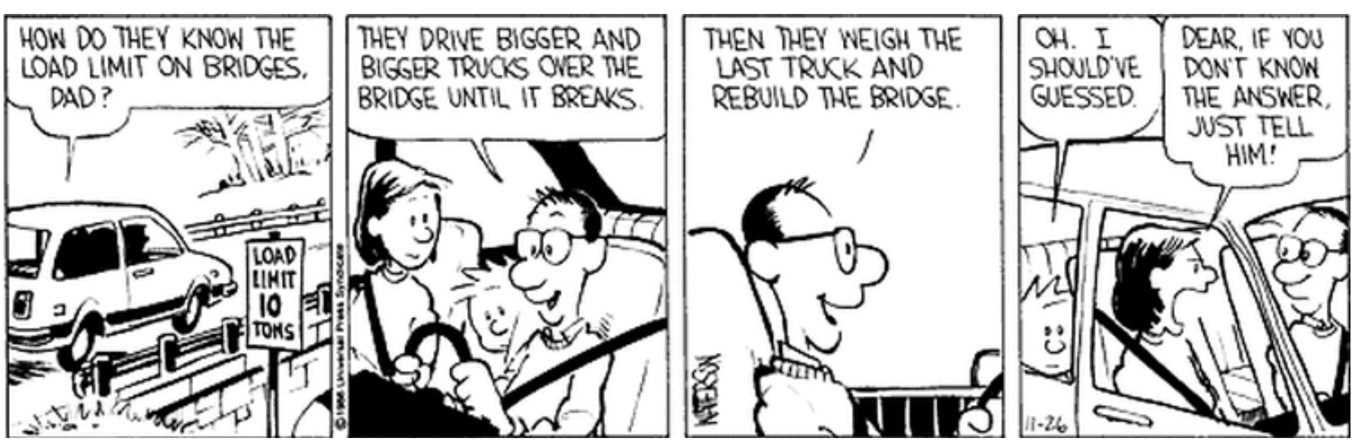
Problem	1 / 2 / 3	4 / 5	6 / 7	8	9	Total
Possible	27	19	26	12	16	100
Received						

DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 x 5 card (both sides) of handwritten notes and a calculator.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.





4 points **1.** In New York City people often take the subway or they take a taxi (taxis and subways are in competition with each other). Let $f(p_1, p_2)$ be the number of people who take the subway, where p_1 is the price of a subway ride and p_2 is the price of a taxi ride. What would be the sign of each of the following? (Just circle one for each derivative.)

$\frac{\partial f}{\partial p_1}$ would be: > 0 $= 0$ < 0 As $p_1 \uparrow$, $f \downarrow$

$\frac{\partial f}{\partial p_2}$ would be: > 0 $= 0$ < 0 As $p_2 \uparrow$, $f \uparrow$

14 points **2.** For $f(x,y) = e^{y/x^2}$, find the following derivatives: $f = e^{yx^{-2}}$

/3 $\frac{\partial f}{\partial x} = e^{yx^{-2}} \cdot y \cdot (-2) x^{-3}$

/3 $\frac{\partial f}{\partial y} = e^{yx^{-2}} \cdot x^{-2}$

/4 $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(e^{yx^{-2}} \cdot x^{-2}\right) = e^{yx^{-2}} \cdot x^{-2} \cdot x^{-2}$

/4 $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(e^{yx^{-2}} \cdot x^{-2}\right) = e^{yx^{-2}} \cdot y(-2)x^{-3} \cdot x^{-2} + e^{yx^{-2}} \cdot (-2)x^{-3}$

using product rule.

9 points **3.** Consider the production function $f(x,y) = 10x^{1/2}y^{1/2}$ where x is units of labor and y is units of capital.

/3 Find and interpret $f(25, 49)$. $= 10\sqrt{25}\sqrt{49} = 350$

Production with 25 units labor, 49 units capital.

/4 Find and interpret $\frac{\partial f}{\partial x}(25, 49)$. $\frac{\partial f}{\partial x} = 10\left(\frac{1}{2}\right)x^{-1/2}y^{1/2} = \frac{5\sqrt{y}}{\sqrt{x}}$

$\frac{\partial f}{\partial x}(25, 49) = \frac{5\sqrt{49}}{\sqrt{25}} = 7$. If $x \uparrow 1$, $f \uparrow 7$.

/2 Find the marginal productivity of labor at $(x, y) = (25, 49)$.

9 points **4.** Suppose for $f(x,y)$ that: $f(1,2) = 20$, $\frac{\partial f}{\partial x}(1,2) = 10$, and $\frac{\partial f}{\partial y}(1,2) = 5$.
Using this info, estimate each of the following:

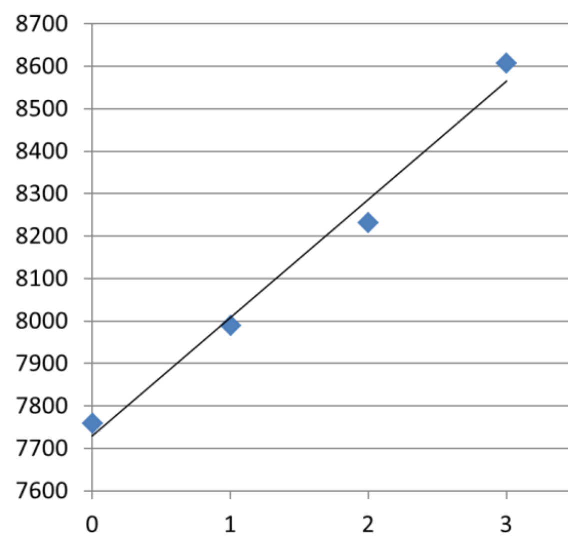
$1+3$ /4 $f(\mathbf{4}, 2) \approx$ $20 + 3 \cdot 10 = 50$

$2+2$ /4 $f(1, \mathbf{4}) \approx$ $20 + 2 \cdot 5 = 30$

/1 $f(\mathbf{4}, \mathbf{4}) \approx$ $20 + 3 \cdot 10 + 2 \cdot 5 = 60$

10 points **5.** Consider the following data which lists the per capita spending on health care in the United States from 2020 to 2023.

Years t after 2020	Spending S
0	\$7760
1	\$8010
2	\$8230
3	\$8600



The line that best fits these data is

$$S = 275t + 7750,$$

where t is the number of years after 2020.

/2 What does the value 275 represent? (Don't just say slope—explain what it tells us.)

If $t \uparrow 1$, then $S \uparrow 275$.
That is, spending increases by \$275/year.

/4 Based on these data (i.e. this line), what do you estimate the per capita spending will be in the year 2030 (10 years after 2020)?

$S = 275(10) + 7750 = \$10,500.$

/4 Based on these data, by approximately **what year** do you expect per capita spending to reach \$15,000?

$15000 = 275t + 7750$

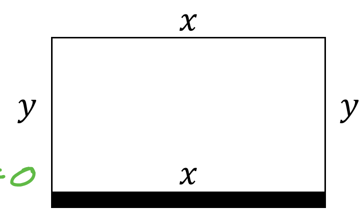
$\Rightarrow t = \dfrac{15000 - 7750}{275} \approx 26.36,$

So year $2020 + 26.36$, i.e. about $\frac{1}{3}$ through year 2046.

13 points **6.** Suppose we are trying to enclose a rectangular garden, where the lower wall is made of stone and costs \$90 per foot, and the upper and two side walls are made of wood at \$30 per foot. So the cost of the four walls would be

$$90x + 30x + 30y + 30y = 120x + 60y. = 1200$$

so $1200 - 120x - 60y = 0$



Find the values of x and y which maximize the area xy given the constraint that we have \$1200 to spend on the wall. Use the Lagrange Multiplier Method.

$F = xy + \lambda(1200 - 120x - 60y)$

$= xy + 1200\lambda - 120\lambda x - 60\lambda y$

$\frac{\partial F}{\partial x} = y - 120\lambda = 0 \Rightarrow \lambda = \frac{y}{120}$

$\frac{\partial F}{\partial y} = x - 60\lambda = 0 \Rightarrow \lambda = \frac{x}{60}$

$\frac{y}{120} = \frac{x}{60}$

$\Rightarrow y = 2x$

In constraint: $120x + 60(2x) = 1200$

$\Rightarrow x = 5$

$y = 10$

Extra credit: by approximately how much would the area increase if we could spend \$1 more on building the wall? Use λ to answer this question.

$\lambda = \frac{10}{120} \left(= \frac{5}{60} \right) = \frac{1}{12}.$

13 points **7.** Same information as in the previous problem, but now we want to find the values of x and y which minimize cost with the constraint that the area be 400 ft^2. Use any Calculus-based method you like—you do not necessarily have to use the Lagrange Multiplier Method (of course you can), but you do need to use Calculus (derivatives) in solving this.

$F = 120x + 60y + \lambda(400 - xy)$

$= 120x + 60y + 400\lambda - \lambda xy$

$\frac{\partial F}{\partial x} = 120 - \lambda y = 0 \Rightarrow \lambda = \frac{120}{y}$

$\frac{\partial F}{\partial y} = 60 - \lambda x = 0 \Rightarrow \lambda = \frac{60}{x}$

$\frac{120}{y} = \frac{60}{x} \Rightarrow y = 2x$

In constraint:

$x(2x) = 400 \Rightarrow x = \sqrt{200}$

$y = 2\sqrt{200}$

12 points **8.** Suppose a monopolist produces and sells a certain product in two different countries.

Country	Number sold	Price
1	x	$90 - \frac{x}{20}$
2	y	$100 - \frac{y}{10}$

Then the total revenue is

$$R(x,y) = x\left(90 - \frac{x}{20}\right) + y\left(100 - \frac{y}{10}\right).$$

If there is a fixed cost of \$5000 and a cost-per-item of \$10, then the total cost is

$$C(x,y) = 5000 + 10(x+y).$$

Then *Profit* = *Revenue* – *Cost*:

$$P(x,y) = x\left(90 - \frac{x}{20}\right) + y\left(100 - \frac{y}{10}\right) - [5000 + 10(x+y)]$$

$$= -\frac{1}{20}x^2 + 80x - \frac{1}{10}y^2 + 90y - 5000$$

So here is the problem: find the values of x and y which maximize profit

$$P(x,y) = -\frac{1}{20}x^2 + 80x - \frac{1}{10}y^2 + 90y - 5000.$$

Use first derivatives to find the values of x and y, and use second derivatives to make sure that you have found where $P(x,y)$ is *maximized* (rather than *minimized*).

$$\frac{\partial P}{\partial x} = -\frac{x}{10} + 80 = 0 \Rightarrow x = 800$$

$$\frac{\partial P}{\partial y} = -\frac{y}{5} + 90 = 0 \Rightarrow y = 450$$

$$\frac{\partial^2 P}{\partial x^2} = -\frac{1}{10}$$

$$\frac{\partial^2 P}{\partial y^2} = -\frac{1}{5}$$

$$\frac{\partial^2 P}{\partial x \partial y} = 0$$

$$\frac{\partial^2 P}{\partial y \partial x} = 0$$

$$D(x,y) = \left(-\frac{1}{10}\right)\left(-\frac{1}{5}\right) - 0^2 = \frac{1}{50}$$

$D(800,450) = \frac{1}{50} > 0$, so min or max.

$\frac{\partial^2 f}{\partial x^2}(800,450) = -\frac{1}{10} < 0$, so max.

Notice prices are:

$$90 - \frac{800}{20} = 50,$$

$$100 - \frac{450}{10} = 55$$

16 points **9.** Suppose now that the monopolist in Problem 8 must charge the same price in both countries. That is, it must be that $90 - \frac{x}{20} = 100 - \frac{y}{10}$.

Find the values of x and y which maximize profit

$$P(x, y) = -\frac{1}{20}x^2 + 80x - \frac{1}{10}y^2 + 90y - 5000.$$

subject to the equal price constraint that $90 - \frac{x}{20} = 100 - \frac{y}{10}$; i.e. $0 = 10 + \frac{x}{20} - \frac{y}{10}$

$$F(x,y) = -\frac{1}{20}x^2 + 80x - \frac{1}{10}y^2 + 90y - 5000 + \lambda\left(10 + \frac{x}{20} - \frac{y}{10}\right)$$

$$= -\frac{1}{20}x^2 + 80x - \frac{1}{10}y^2 + 90y - 5000 + 10\lambda + \frac{\lambda}{20}x - \frac{\lambda}{10}y$$

$$\frac{\partial F}{\partial x} = -\frac{x}{10} + 80 + \frac{\lambda}{20} = 0 \Rightarrow \lambda = 20\left(\frac{x}{10} - 80\right) = 2x - 1600$$

$$\frac{\partial F}{\partial y} = \frac{-y}{5} + 90 - \frac{\lambda}{10} = 0 \Rightarrow \lambda = 10\left(\frac{-y}{5} + 90\right) = -2y + 900$$

So $2x - 1600 = -2y + 900 \Rightarrow x = 1250 - y$

In constraint:

$$\left(0 = 10 + \frac{1250-y}{20} - \frac{y}{10}\right) \cdot 20$$

$$\Rightarrow 0 = 200 + 1250 - y - 2y$$

$$\Rightarrow y = \frac{1450}{3} = 483\tfrac{1}{3}$$

$$x = 1250 - \swarrow = 766\tfrac{2}{3}$$

Notice prices:

$$\left.\begin{aligned} 90 - \frac{766\frac{2}{3}}{20} &= 51\tfrac{2}{3} \\ 100 - \frac{483\frac{1}{3}}{10} &= 51\tfrac{2}{3} \end{aligned}\right\}$$

Same, and between original prices of 50 and 55.