Math 316

Section 3.2 Introduction to Probability
Section 3.3 Probability Rules

I don't like the way the book divides the ideas of 3.2 and 3.3 into these two sections, so we'll cover both at the same time. In class let's bet:

1. Fifteen people choose a number between 1 and 100. Will there be a match? Repeat?
2. Do two or more people in our class share the same birthday? (I've not looked this up, so I don't know the answer to this question until we ask everyone in class.)

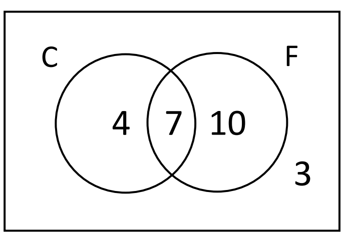
The probability of an event E occurring is what fraction of the time we expect it to occur. The theoretical probability of event E occurring is

$$\Pr\{E\} = \frac{Number\ of\ ways\ E\ can\ occur}{Number\ of\ possible\ outcomes}.$$

Related to this is conditional probability, $\Pr\{E_2|E_1\}$: the probability that event E_2 will occur or be true, given that event E_1 has occurred or is true.

$$\Pr\{E_2|E_1\} = \frac{\boldsymbol{With\ E_1\ true},\ number\ of\ ways\ E_2\ can\ occur}{\boldsymbol{With\ E_1\ true},\ number\ of\ possible\ outcomes}$$

Example: Students (from a previous semester) from **C**alifornia (or not) and/or **F**emale (or not):



$Pr\{C\} = \frac{11}{24}$ $\quad$ $Pr\{C^C\} = \frac{13}{24}$

$Pr\{F\} = \frac{17}{24}$ $\quad$ $Pr\{F^C\} = \frac{7}{24}$

$Pr\{C\ and\ F\} = \frac{7}{24}$ $\quad$ $Pr\{C\ or\ F\} = \frac{21}{24}$

$Pr\{C|F\} = \frac{7}{17}$ $\quad$ $Pr\{C^C|F\} = \frac{10}{17}$

$Pr\{C|F^C\} = \frac{4}{7}$ $\quad$ $Pr\{C^C|F^C\} = \frac{3}{7}$

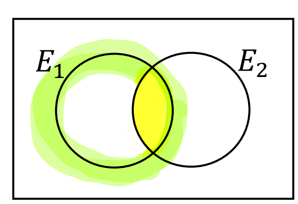
$Pr\{F|C\} = \frac{7}{11}$ $\quad$ $Pr\{F^C|C\} = \frac{4}{11}$

In this example we've used the fact that:

$$\Pr\{E_2|E_1\} = \frac{\boldsymbol{With\ E_1\ true},\ number\ of\ ways\ E_2\ can\ occur}{\boldsymbol{With\ E_1\ true},\ number\ of\ possible\ outcomes}$$

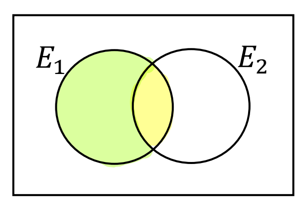
$$= \frac{Number\ of\ ways\ both\ E_1\ and\ E_2\ can\ occur}{Number\ ways\ E_1\ can\ occur}$$

$$= \frac{n\{E_1 and\ E_2\}}{n\{E_1\}}.$$



Eight rules/ideas/facts about probabilities.

1. The probability of any event E occurring is between 0% and 100%; that is,
$$0 \leq \Pr\{E\} \leq 1.$$
2. There is a probability of 1 (that is, a 100% chance) that one of the possible outcomes will occur.
3. There is a probability of 1 (100% chance) that an event will either occur or not occur. That is, where E^C ("E complement," i.e. "the complement of E") is the event that E does *not* occur, then $\Pr\{E\} + \Pr\{E^C\} = 1$. An important consequence of this is that $\Pr\{E^C\} = 1 - \Pr\{E\}$ and
$$\Pr\{E\} = 1 - \Pr\{E^C\}.$$
4. If events E_1 and E_2 are disjoint (that is, they are mutually exclusive, that is, they can't both occur simultaneously), then $\Pr\{E_1 \text{ or } E_2\} = \Pr\{E_1\} + \Pr\{E_2\}$.
5. In general,
$$\Pr\{E_1 \text{ or } E_2\} = \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1 \text{ and } E_2\}.$$
6. If two events are independent (knowing whether one is true or has occurred does *not* affect how likely the other is to be true or occur), then
$$\Pr\{E_1 \mid E_2\} = \Pr\{E_1\}$$
$$\Pr\{E_2 \mid E_1\} = \Pr\{E_2\}$$
$$\Pr\{E_1 \text{ and } E_2\} = \Pr\{E_1\} \cdot \Pr\{E_2\}.$$
7. In general,
$$\Pr\{E_1 \text{ and } E_2\} = \Pr\{E_1\} \cdot \Pr\{E_2 | E_1\} = \Pr\{E_2\} \cdot \Pr\{E_1 | E_2\}$$
8. Given two events, E_1 and E_2, there are two ways E_1 can occur: when E_2 is true or occurs, or when E_2 is not true or does not occur. Thus:
$$\Pr\{E_1\} = \Pr\{(E_1 \text{ and } E_2) \text{ or } (E_1 \text{ and } E_2^C)\}$$
$$= \Pr\{E_1 \text{ and } E_2\} + \Pr\{E_1 \text{ and } E_2^C\}$$
$$= \Pr\{E_2\} \cdot \Pr\{E_1 | E_2\} + \Pr\{E_2^C\} \cdot \Pr\{E_1 | E_2^C\}$$



Some observations and notes on these rules.

Regarding Rule 1. Question: why use a decimal value between 0 and 1 rather than a percentage? Answer: They are the same thing, silly!. "Per cent" simple means "per 100," so, for example, $70\% = 70/100 = 0.70$.

In class two examples regarding the usefulness of Rule 3. **Class Example 1: choose a number between 1 and 100**. **Class Example 2: birthdays**. **Online: Birthday probabilities**.

Rule 4 is a special case of Rule 5, the case in which E_1 and E_2 are disjoint (mutually exclusive). **Figures 3.3.1 and 3.3.2**. Also notice **in the California and/or Female example on the front page** that

$$Pr\{C \text{ } or \text{ } F\} = Pr\{C\} + Pr\{F\} - Pr\{C \text{ } and \text{ } F\}.$$

See the definition for <u>conditional probability</u> on page 96. Where this formula comes from:

$$\begin{aligned}\Pr\{E_2|E_1\} &= \frac{\boldsymbol{With\ E_1\ true},\ the\ number\ of\ ways\ E_2\ can\ occur}{\boldsymbol{With\ E_1\ true},\ the\ number\ of\ possible\ outcomes} \\ &= \frac{Number\ of\ ways\ both\ E_1\ and\ E_2\ can\ occur}{Number\ ways\ E_1\ can\ occur} \\ &= \frac{Number\ of\ ways\ both\ E_1\ and\ E_2\ can\ occur\ \ /\ \ Number\ of\ possible\ outcomes}{Number\ ways\ E_1\ can\ occur\ \ /\ \ Number\ of\ possible\ outcomes} \\ &= \frac{\Pr\{E_1 and\ E_2\}}{\Pr\{E_1\}}.\end{aligned}$$

This is essentially identical to the formula we already had: $\Pr\{E_2|E_1\} = \frac{n\{E_1 and\ E_2\}}{n\{E_1\}}$.

This other version of $\Pr\{E_2|E_1\}$ leads to Rule 7: $\Pr\{E_1\ and\ E_2\} = \Pr\{E_1\} \cdot \Pr\{E_2|E_1\}$. **Class Example 3: select a ball from a bin**. Rules 7 and 8 both show up in **Example 3.2.11**, **Figure 3.2.5**. Let's next work **Class Example 4: medical testing**, a variation of **Example 3.2.11**.

As seen in Class Example 3 and Book Example 3.2.11, <u>probability trees</u> help us visualize the possible outcomes and their probabilities. If an event can occur in multiple ways, then the probability of that event occurring is the sum of the probabilities of each of those ways the event can occur, as we saw in Class Example 3. Notice: the sum of each set of branches is 1 (this fact is a good way to double check your numbers).

Rule 6 is a special case of Rule 7. *Events* E_1 *and* E_2 *are independent* means that knowing that one has occurred does not affect how likely the other is to occur. If E_2 is independent of E_1, then $\Pr\{E_2|E_1\} = \Pr\{E_2\}$ so that

$$\Pr\{E_1 \; and \; E_2\} = \Pr\{E_1\} \cdot \Pr\{E_2|E_1\} = \Pr\{E_1\} \cdot \Pr\{E_2\}\,.$$

It turns out that if E_2 is independent of E_1, then E_1 is independent of E_2, that is, E_1 and E_2 are independent of each other.

So how do you determine if two events are independent? We can check if any one of the three conditions is true:

$$\Pr\{E_1 \; and \; E_2\} = \Pr\{E_1\} \cdot \Pr\{E_2\}$$

$$\Pr\{E_1|E_2\} = \Pr\{E_1\}$$

$$\Pr\{E_2|E_1\} = \Pr\{E_2\}$$

If any one of them is true, all three are true, and if any one of them is not true, all three are not true. So we can check any one of the conditions to determine independence. Let's work **HW 3.3.5**. And **HW 3.3.1** if time.

A few final thoughts. Some probabilities are very intuitive and what you would expect, while others can be surprising. You should use your intuition as a guide, but ultimately trust the formulas. We saw this in Class Example 4. Also, **Online**: **Mammogram Articles**. You can experiment with the values in **Example 3.2.11** using with **Online: Example 3.2.11.**

While the outcome of one (or even a few experiments) have exactly the probability one might expect, the more you repeat the experiment, the closer the outcomes will match what theory says should happen. **Example 3.2.6**. Where does the 0.58 probability come from? $\Pr(both\ black\} = (.3)(.3) = .09$ and $\Pr\{both\ gray\} = (.7)(.7) = .49$. These add up to $.58$.