Math 316

Section 10.3 Independence and Association in 2×2 Contingency Table

Recall notation for conditional probabilities: $\Pr\{A|B\}$ is the probability that A is or will be true if B is true.

Here is Book Table 10.5.4. Section 10.3 discusses this sort of problem, but only for the 2 x 2 case. We can have the same discussion, but we might as well do so in a more general way (for larger than 2 x 2 contingency tables).

		Hair Color				
		Brown	*Black*	*Fair*	*Red*	**Total**
Eye Color	*Brown*	438	288	115	16	**857**
	Grey/Green	1,387	746	946	53	**3,132**
	Blue	807	189	1,768	47	**2,811**
	Total	**2,632**	**1,223**	**2,829**	**116**	**6,800**

Some probabilities about Brown Eyes and Hair Colors:

$$\widehat{\Pr}\{Brown\ Eyes\} = \frac{857}{6800} \approx .126$$

$\widehat{\Pr}\{Brown\ Eyes \mid Brown\ Hair\} = \frac{438}{2632} \approx .166$ $\quad$ $\widehat{\Pr}\{Brown\ Hair\} = \frac{2632}{6800} \approx .387$

$\widehat{\Pr}\{Brown\ Eyes \mid Black\ Hair\} = \frac{288}{1223} \approx .235$ $\quad$ $\widehat{\Pr}\{Black\ Hair\} = \frac{1223}{6800} \approx .180$

$\widehat{\Pr}\{Brown\ Eyes \mid Fair\ Hair\} = \frac{115}{2829} \approx .041$ $\quad$ $\widehat{\Pr}\{Fair\ Hair\} = \frac{2829}{6800} \approx .416$

$\widehat{\Pr}\{Brown\ Eyes \mid Red\ Hair\} = \frac{16}{116} \approx .138$ $\quad$ $\widehat{\Pr}\{Red\ Hair\} = \frac{116}{6800} \approx .017$

$$\widehat{\Pr}\{Brown\ H. and\ Brown\ E.\} = \widehat{\Pr}\{Brown\ H.\} \cdot \widehat{\Pr}\{Brown\ E. \mid Brown\ H.\} = \left(\frac{2632}{6800}\right)\left(\frac{438}{2632}\right) \approx (.387)(.166) \approx .058$$

$$\widehat{\Pr}\{Black\ H. and\ Brown\ E.\} = \widehat{\Pr}\{Black\ H.\} \cdot \widehat{\Pr}\{Brown\ E. \mid Black\ H.\} = \left(\frac{1223}{6800}\right)\left(\frac{288}{1223}\right) \approx (.180)(.235) \approx .042$$

$$\widehat{\Pr}\{Fair\ H. and\ Brown\ E.\} = \widehat{\Pr}\{Fair\ H.\} \cdot \widehat{\Pr}\{Brown\ E. \mid Fair\ H.\} = \left(\frac{2829}{6800}\right)\left(\frac{115}{2829}\right) \approx (.416)(.041) \approx .017$$

$$\widehat{\Pr}\{Red\ H. and\ Brown\ E.\} = \widehat{\Pr}\{Red\ H.\} \cdot \widehat{\Pr}\{Brown\ E. \mid Red\ H.\} = \left(\frac{116}{6800}\right)\left(\frac{16}{116}\right) \approx (.017)(.138) \approx .002$$

Notice for Brown Eyes that

$$\min\{.166, .235, .041, .138\} < .126 < \max\{.166, .235, .041, .138\}.$$

Indeed, one way to think of $.126$ is that it is a weighted average of $.166, .235, .041, .138$, the proportions of each hair color with brown eyes, where the weightings are the fraction of the entire population each hair color group is (again, this is for Brown Eyes):

All hair colors		Brown hair		Black hair		Fair hair		Red hair
$\mathbf{.126}$	$=$	$\mathbf{.166}\ (.387)$	$+$	$\mathbf{.235}\ (.180)$	$+$	$\mathbf{.041}\ (.416)$	$+$	$\mathbf{.138}\ (.017)$
	$\approx$	$.058$	$+$	$.042$	$+$	$.017$	$+$	$.002$

Two probabilities about Fair Hair: And two about Black Hair:

$\widehat{\Pr}\{Fair\ Hair\} = \frac{2829}{6800} \approx .416$ $\widehat{\Pr}\{Black\ Hair\} = \frac{1223}{6800} \approx .180$

$\widehat{\Pr}\{Fair\ Hair \mid Brown\ Eyes\} = \frac{115}{857} \approx .134$ $\widehat{\Pr}\{Black\ Hair \mid Brown\ Eyes\} = \frac{288}{857} \approx .336$

Next, see **Observation 1** about these probabilities listed on the front side of this sheet. We see that Brown Eyes and Black Hair seem to "go together"—they are associated with each other: if one is true, the other is *more* likely to be true. (This is a different than causality, where one being true causes the other to be true. For example, if may be that whatever is causing the one to be true is also causing the other to be true as well.) And Brown Eyes and Fair Hair have the opposite relationship: if one is true, the other is *less* likely to be true. What we've seen in this example is what the book is attempting to describe with Facts 10.3.1 and 10.3.2 and Examples 10.3.4 and 10.3.5.

Recall that if $\Pr\{A|B\} = \Pr\{A\}$, then A and B are said to be independent.

It turns out the connections see in Observation 1 are even stronger than just described. It's not just the sign/direction of the change, but also the amount. See **Observation 2**. So we see that a person who is Fair Haired is nearly twice as likely (as a person whose hair color is unknown, i.e. the general population) to have Brown Eyes, and conversely.

What we've just observed can be pretty easily proved, which we do in **Observation 3**.

A final thought in **Observation 4**. So people with Black Hair are about 5.7 times as likely to have Brown Eyes as people with Fair Hair.

Section 10.6 Applicability of Methods

Read this on your own. If there is time, we'll look at the **Conditions for Validity** at the beginning of Section 10.6. In general, the more information we have (and assuming we are correctly using that information), the better our hypothesis testing will be. On your own, see Book Example 10.6.3 as an example of this. Some of the 10.6 homework problems will leave you leaving a little uncertain, so do your best on those problems and then take a look at the solutions afterwards.