Sets and Counting

In this chapter, we introduce some ideas useful in the study of probability. Our first topic, the theory of sets, will provide a convenient language and notation in which to discuss probability. Using set theory, we develop a number of counting principles that can also be applied to computing probabilities.

1 Sets

In many applied problems, one must consider collections of various sorts of objects. For example, a survey of unemployment trends might consider the collection of all U.S. cities with current unemployment greater than 9 percent. A study of birthrates might consider the collection of countries with a current birthrate less than 20 per 1000 population. Such collections are examples of sets. A **set** is any collection of objects. The objects, which may be countries, cities, years, numbers, letters, or anything else, are called the **elements** of the set. A set is often specified by listing its elements inside a pair of braces. For example, the set whose elements are the first six letters of the alphabet is written

$$\{a, b, c, d, e, f\}.$$

Similarly, the set whose elements are the even numbers between 1 and 11 is written

$$\{2, 4, 6, 8, 10\}.$$

We can also specify a set by giving a description of its elements (without actually listing the elements). For example, the set $\{a, b, c, d, e, f\}$ can also be written

$$\{\text{the first six letters of the alphabet}\},$$

and the set $\{2, 4, 6, 8, 10\}$ can be written

$$\{\text{all even numbers between 1 and 11}\}.$$

For convenience, we usually denote sets by capital letters, A, B, C, and so on.

Two sets A and B are said to be **equal** if every element of A is also in B, and every element of B is also in A. For example,

$$\{a, b, c, d, e, f\} = \{f, e, d, b, c, a\}$$

and

$$\{2, 4, 6, 8\} = \{2, 2, 4, 4, 4, 6, 6, 8\}.$$

The great diversity of sets is illustrated by the following examples:

1. Let $C = \{\text{possible sequences of outcomes of tossing a coin three times}\}$. If we let H denote "heads" and T denote "tails," the various sequences can be easily described:

$$C = \{\text{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT}\},$$

where, for instance, THH means "first toss tails, second toss heads, third toss heads."

2. Let $B = \{\text{license plate numbers consisting of three letters followed by three digits}\}$. Some elements of B are

SBG 602, GXZ 179, YHJ 006.

The number of elements in B is sufficiently large so that listing all of them is impractical. However, in this chapter, we develop a technique that allows us to calculate the number of elements of B.

3. The graph of the equation $y = x^2$ is the set of all points (a, b) in the plane for which $b = a^2$. This set has infinitely many elements.

Sets arise in many practical contexts, as the next example shows.

EXAMPLE 1

Listing the Elements of a Set Table 1 gives the rate of inflation, as measured by the percentage change in the consumer price index, for the years from 1996 to 2015. Let

$$A = \{\text{years from 1996 to 2015 in which inflation was above 3\%}\}$$
$$B = \{\text{years from 1996 to 2015 in which inflation was below 2\%}\}.$$

Determine the elements of A and B.

Table 1 **U.S. Inflation Rates**

Year	Inflation (%)	Year	Inflation (%)
1996	3.0	2006	3.2
1997	2.3	2007	2.8
1998	1.6	2008	3.8
1999	2.2	2009	−0.4
2000	3.4	2010	1.6
2001	2.8	2011	3.2
2002	1.6	2012	2.1
2003	2.3	2013	1.5
2004	2.7	2014	1.6
2005	3.4	2015	0.1

(Source: www.bls.gov)

SOLUTION By reading Table 1, we see that

$$A = \{2000, 2005, 2006, 2008, 2011\}$$
$$B = \{1998, 2002, 2009, 2010, 2013, 2014, 2015\}.$$

>> Now Try Exercises 9(a) and (b)

Suppose that we are given two sets, A and B. Then it is possible to form new sets from A and B.

DEFINITIONS Union and Intersection of Two Sets The **union** of A and B, written $A \cup B$ and pronounced "A union B," is defined as follows:

$A \cup B$ is the set of all elements that belong to either A or B (or both).

The **intersection** of A and B, written $A \cap B$ and pronounced "A intersection B," is defined as follows:

$A \cap B$ is the set of all elements that belong to both A and B.

For example, let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5, 7, 11\}$. Then,

$$A \cup B = \{1, 2, 3, 4, 5, 7, 11\}$$
$$A \cap B = \{1, 3\}.$$

EXAMPLE 2

The Intersection and Union of Sets Table 2 gives the rates of unemployment and inflation for the years from 2001 to 2015. Let

$$A = \{\text{years from 2001 to 2015 in which unemployment is at least 5\%}\}$$
$$B = \{\text{years from 2001 to 2015 in which the inflation rate is at least 3\%}\}.$$

(a) Describe the sets $A \cap B$ and $A \cup B$.
(b) Determine the elements of A, B, $A \cap B$, and $A \cup B$.

Table 2 U.S. Unemployment and Inflation Rates

Year	Unemployment (%)	Inflation (%)
2001	4.7	2.8
2002	5.8	1.6
2003	6.0	2.3
2004	5.5	2.7
2005	5.1	3.4
2006	4.6	3.2
2007	4.6	2.8
2008	5.8	3.8
2009	9.3	−0.4
2010	9.6	1.6
2011	8.9	3.2
2012	8.1	2.1
2013	7.4	1.5
2014	6.2	1.6
2015	5.3	0.1

(Source: www.bls.gov)

SOLUTION **(a)** From the descriptions of A and B, we have

$$A \cap B = \{\text{years from 2001 to 2015 in which unemployment is at least 5\% and inflation is at least 3\%}\}$$

$$A \cup B = \{\text{years from 2001 to 2015 in which either unemployment is at least 5\% or inflation is at least 3\% (or both)}\}.$$

(b) From the table, we see that

$$A = \{2002, 2003, 2004, 2005, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015\}$$

$$B = \{2005, 2006, 2008, 2011\}$$

$$A \cap B = \{2005, 2008, 2011\}$$

$$A \cup B = \{2002, 2003, 2004, 2005, 2006, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015\}.$$

» Now Try Exercises 9(c) and (d)

We have defined the union and the intersection of two sets. In a similar manner, we can define the union and intersection of any number of sets. For example, if A, B, and C are three sets, then their union, denoted $A \cup B \cup C$, is the set whose elements are precisely those that belong to at least one of the sets A, B, and C. Similarly, the intersection of A, B, and C, denoted $A \cap B \cap C$, is the set consisting of those elements that belong to all of the sets A, B, and C. In a similar way, we may define the union and intersection of more than three sets.

Suppose that we are given a set A. We may form new sets by selecting elements from A. Sets formed in this way are called *subsets* of A.

> **DEFINITION Subset of a Set** The set B is a **subset** of the set A, written $B \subseteq A$ and pronounced "B is a subset of A," provided that every element of B is an element of A.

For example, $\{1, 3\} \subseteq \{1, 2, 3\}$.

One set that is considered very often is the set that contains no elements at all. This set is called the **empty set** (or *null set*) and is written $\varnothing$ or $\{\ \}$. The empty set is a subset of every set. (Here is why: Let A be any set. Every element of $\varnothing$ also belongs to A. If you do not agree, then you must produce an element of $\varnothing$ that does not belong to A. But you cannot, since $\varnothing$ has no elements. So $\varnothing \subseteq A$.)

> **DEFINITION Disjoint Sets** Two sets A and B are **disjoint** if they have no elements in common, that is, if $A \cap B = \varnothing$.

EXAMPLE 3 Listing the Subsets of a Set

Let $A = \{a, b, c\}$. Find all subsets of A.

SOLUTION Since A contains three elements, every subset of A has at most three elements. We look for subsets according to the number of elements:

Number of Elements in Subset	Possible Subsets
0	$\varnothing$
1	$\{a\}, \{b\}, \{c\}$
2	$\{a, b\}, \{a, c\}, \{b, c\}$
3	$\{a, b, c\}$

Thus, we see that A has eight subsets, namely, those listed on the right. (Note that we count A as a subset of itself.)

» Now Try Exercise 5

In general, if a set A contains n elements, then A will have 2^n subsets. (In Section 7, we will prove this fact.)

It is usually convenient to regard all sets involved in a particular discussion as subsets of a single larger set. Thus, for example, if a problem involves the sets $\{a, b, c\}$, $\{e, f\}$, $\{g\}$, $\{b, x, y\}$, then we can regard all of these as subsets of the set

$$U = \{\text{all letters of the alphabet}\}.$$

Since U contains all elements being discussed, it is called a **universal set** (for the particular problem). In this text, we shall specify the particular universal set that we have in mind or it will be clearly defined by the context.

The set A contained in the universal set U has a counterpart, called its *complement*.

DEFINITION Complement of a Set The **complement** of A, written A' and pronounced "A complement," is defined as follows:

A' is the set of all elements in the universal set U that do not belong to A.

For example, let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{2, 4, 6, 8\}$. Then,

$$A' = \{1, 3, 5, 7, 9\}.$$

EXAMPLE 4 **Finding the Complement of a Set** Let $U = \{a, b, c, d, e, f, g\}$, $S = \{a, b, c\}$, and $T = \{a, c, d\}$. List the elements of the following sets:

(a) S' **(b)** T' **(c)** $(S \cap T)'$ **(d)** $S' \cap T'$ **(e)** $S' \cup T'$

SOLUTION

(a) S' consists of those elements of U that are not in S, so $S' = \{d, e, f, g\}$.

(b) Similarly, $T' = \{b, e, f, g\}$.

(c) As is the case in arithmetic, we perform the operation in parentheses first. To determine $(S \cap T)'$, we must first determine $S \cap T$:

$$S \cap T = \{a, c\}.$$

Then, we determine the complement of this set:

$$(S \cap T)' = \{b, d, e, f, g\}.$$

(d) We determined S' and T' in parts (a) and (b). The set $S' \cap T'$ consists of the elements that belong to both S' and T'. Therefore, referring to parts (a) and (b), we have

$$S' \cap T' = \{e, f, g\}.$$

(e) Since $S' \cup T'$ consists of the elements that belong to S' or T' (or both),

$$S' \cup T' = \{b, d, e, f, g\}.$$

>> *Now Try Exercises 9(e) and (f)*

NOTE The results of parts (c) and (d) show that in general $(S \cap T)'$ *is not* the same as $S' \cap T'$. In the next section, we show that $(S \cap T)'$ *is* the same as $S' \cup T'$. «

EXAMPLE 5 **College Students** Let $U = \{$students at Gotham College$\}$, $E = \{$students at Gotham College who are at most 18 years old$\}$, and $S = \{$STEM majors at Gotham College$\}$.

(a) Use set-theoretic notation (that is, union, intersection, and complement symbols) to describe {students at Gotham College who are at most 18 years old or are not STEM majors}.

(b) Use set-theoretic notation to describe {students at Gotham College who are older than 18 and are STEM majors}.

(c) Describe in words the set $E' \cap S'$.

(d) Describe in words the set $E' \cup S$.

SOLUTION

(a) $E \cup S'$

(b) $E' \cap S$

(c) {students at Gotham College who are older than 18 and are not STEM majors}

(d) {students at Gotham College who are older than 18 or are STEM majors}

>> Now Try Exercises 21 and 33

The symbol $\in$ is commonly used with sets as a shorthand for "is an element of." For instance, if $S = \{H, T\}$, then $H \in S$. The symbol $\in$ should not be confused with the symbol $\subseteq$. For instance, $H \in S$, but $\{H\} \subseteq S$. The symbol $\notin$ is shorthand for "is not an element of." The symbol $\in$ can be used when defining union, intersection, complement, and subset.

DEFINITIONS

Union	$A \cup B$ is the set of all x such that $x \in A$ or $x \in B$.
Intersection	$A \cap B$ is the set of all x such that $x \in A$ and $x \in B$.
Complement	A' is the set of all $x \in U$ such that $x \notin A$.
Subset	$B \subseteq A$ if $x \in A$ whenever $x \in B$.

Check Your Understanding 1

Solutions can be found following the section exercises.

1. Let $U = \{a, b, c, d, e, f, g\}$, $R = \{a, b, c, d\}$, $S = \{c, d, e\}$, and $T = \{c, e, g\}$. List the elements of the following sets:
   (a) R' (b) $R \cap S$
   (c) $(R \cap S) \cap T$ (d) $R \cap (S \cap T)$
2. Let U = {all Nobel winners}, W = {women who have received Nobel Prizes}, A = {Americans who have received Nobel Prizes}, L = {Nobel winners in literature}. Describe the following sets:
   (a) W' (b) $A \cap L'$ (c) $W \cap A \cap L'$
3. Refer to Problem 2. Use set-theoretic notation to describe {Nobel winners who are American men or recipients of the Nobel Prize in literature}.

EXERCISES 1

1. Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $S = \{1, 2, 3, 4\}$, and $T = \{1, 3, 5, 7\}$. List the elements of the following sets:
   (a) S' (b) $S \cup T$ (c) $S \cap T$ (d) $S' \cap T$
2. Let $U = \{1, 2, 3, 4, 5\}$, $S = \{1, 2, 3\}$, and $T = \{5\}$. List the elements of the following sets:
   (a) S' (b) $S \cup T$ (c) $S \cap T$ (d) $S' \cap T$
3. Let U = {all letters of the alphabet}, $R = \{a, b, c\}$, $S = \{a, e, i, o, u\}$, and $T = \{x, y, z\}$. List the elements of the following sets:
   (a) $R \cup S$ (b) $R \cap S$ (c) $S \cap T$ (d) $S' \cap R$
4. Let $U = \{a, b, c, d, e, f, g\}$, $R = \{a\}$, $S = \{a, b\}$, and $T = \{b, d, e, f, g\}$. List the elements of the following sets:
   (a) $R \cup S$ (b) $R \cap S$ (c) T' (d) $T' \cup S$
5. List all subsets of the set $\{1, 2\}$.
6. List all subsets of the set $\{1, 2, 3, 4\}$.
7. **College Students** Let U = {all college students}, F = {all freshman college students), and B = {all college students who like basketball}. Describe the elements of the following sets:
   (a) $F \cap B$ (b) B' (c) $F' \cap B'$ (d) $F \cup B$
8. **Corporations** Let U = {all corporations}, S = {all corporations with headquarters in New York City}, and T = {all privately owned corporations}. Describe the elements of the following sets:
   (a) S' (b) T' (c) $S \cap T$ (d) $S \cap T'$
9. **S&P Index** The Standard and Poor's Index measures the price of a certain collection of 500 stocks. Table 3 on the next page compares the percentage change in the index during the first 5 business days of certain years with the percentage change for the entire year. Let U = {all years from 1996 to 2015}, S = {all years during which the index increased by 2% or more during the first 5 business days}, and T = {all years for which the index increased by 16% or more during the entire year}. List the elements of the following sets:
   (a) S (b) T (c) $S \cap T$
   (d) $S \cup T$ (e) $S' \cap T$ (f) $S \cap T'$
10. **S&P Index** Refer to Table 3 on the next page. Let U = {all years from 1996 to 2015}, A = {all years during which the index declined during the first 5 business days}, and B = {all years during which the index declined for the entire year}. List the elements of the following sets:
    (a) A (b) B (c) $A \cap B$
    (d) $A' \cap B$ (e) $A \cap B'$
11. **S&P Index** Refer to Exercise 9. Describe in words the fact that $S \cap T'$ has two elements.
12. **S&P Index** Refer to Exercise 10. Describe in words the fact that $A' \cap B$ has two elements.
13. Let $U = \{a, b, c, d, e, f\}$, $R = \{a, b, c\}$, $S = \{a, b, d\}$, and $T = \{e, f\}$. List the elements of the following sets:
    (a) $(R \cup S)'$ (b) $R \cup S \cup T$
    (c) $R \cap S \cap T$ (d) $R \cap S \cap T'$
    (e) $R' \cap S \cap T$ (f) $S \cup T$
    (g) $(R \cup S) \cap (R \cup T)$ (h) $(R \cap S) \cup (R \cap T)$
    (i) $R' \cap T'$

Table 3 Percentage Change in the Standard and Poor's Index

Year	Percent Change for First 5 Days	Percent Change for Year	Year	Percent Change for First 5 Days	Percent Change for Year
2015	0.0	−0.7	2005	−2.1	3.0
2014	−0.5	11.4	2004	1.8	9.0
2013	2.2	29.6	2003	3.4	26.4
2012	1.7	13.4	2002	1.1	−23.4
2011	1.1	0.0	2001	−1.9	−13.0
2010	2.7	12.8	2000	−1.9	−10.1
2009	0.7	23.5	1999	3.7	19.5
2008	−5.3	−38.5	1998	−1.5	26.7
2007	−0.4	4.8	1997	1.0	31.0
2006	3.0	13.6	1996	0.4	20.3

(Source: www.forecast-chart.com, www.investing.com)

14. Let $U = \{1, 2, 3, 4, 5\}$, $R = \{1, 3, 5\}$, $S = \{3, 4, 5\}$, and $T = \{2, 4\}$. List the elements of the following sets:
(a) $R \cap S \cap T$ (b) $R \cap S \cap T'$ (c) $R \cap S' \cap T$
(d) $R' \cap T$ (e) $R \cup S$ (f) $R' \cup R$
(g) $(S \cap T)'$ (h) $S' \cup T'$

In Exercises 15–20, simplify each given expression.

15. $(S')'$ **16.** $S \cap S'$ **17.** $S \cup S'$
18. $S \cap \varnothing$ **19.** $T \cap S \cap T'$ **20.** $S \cup \varnothing$

Corporation A large corporation classifies its many divisions by their performance in the preceding year. Let $P = \{$divisions that made a profit$\}$, $L = \{$divisions that had an increase in labor costs$\}$, and $T = \{$divisions whose total revenue increased$\}$. Describe the sets in Exercises 21–26 by using set-theoretic notation.

21. {divisions that had increases in labor costs or total revenue}

22. {divisions that did not make a profit}

23. {divisions that made a profit despite an increase in labor costs}

24. {divisions that had an increase in labor costs and either were unprofitable or did not increase their total revenue}

25. {profitable divisions with increases in labor costs and total revenue}

26. {divisions that were unprofitable or did not have increases in either labor costs or total revenue}

Automobile Insurance An automobile insurance company classifies applicants by their driving records for the previous three years. Let $S = \{$applicants who have received speeding tickets$\}$, $A = \{$applicants who have caused accidents$\}$, and $D = \{$applicants who have been arrested for driving while intoxicated$\}$. Describe the sets in Exercises 27–32 by using set-theoretic notation.

27. {applicants who have not received speeding tickets}

28. {applicants who have caused accidents and been arrested for drunk driving}

29. {applicants who have received speeding tickets, caused accidents, or been arrested for drunk driving}

30. {applicants who have not been arrested for drunk driving, but have received speeding tickets or have caused accidents}

31. {applicants who have not both caused accidents and received speeding tickets but who have been arrested for drunk driving}

32. {applicants who have not caused accidents or have not been arrested for drunk driving}

College Teachers and Students Let $U = \{$people at Mount College$\}$, $A = \{$students at Mount College$\}$, $B = \{$teachers at Mount College$\}$, $C = \{$people at Mount College who are older than 35$\}$, and $D = \{$people at Mount College who are younger than 35$\}$. Describe verbally the sets in Exercises 33–40.

33. $A \cap D$ **34.** $B \cap C$ **35.** $A \cap B$ **36.** $B \cup C$
37. $A \cup C'$ **38.** $(A \cap D)'$ **39.** D' **40.** $D \cap U$

Ice Cream Preferences Let $U = \{$all people$\}$, $S = \{$people who like strawberry ice cream$\}$, $V = \{$people who like vanilla ice cream$\}$, and $C = \{$people who like chocolate ice cream$\}$. Describe the sets in Exercises 41–46 by using set-theoretic notation.

41. {people who don't like vanilla ice cream}

42. {people who like vanilla but not chocolate ice cream}

43. {people who like vanilla but not chocolate or strawberry ice cream}

44. {people who don't like any of the three flavors of ice cream}

45. {people who like neither chocolate nor vanilla ice cream}

46. {people who like only strawberry and chocolate ice cream}

47. Let U be the set of vertices in Fig. 1. Let $R = \{$vertices (x, y) with $x > 0\}$, $S = \{$vertices (x, y) with $y > 0\}$, and $T = \{$vertices (x, y) with $x \leq y\}$. List the elements of the following sets:
(a) R (b) S (c) T
(d) $R' \cup S$ (e) $R' \cap T$ (f) $R \cap S \cap T$

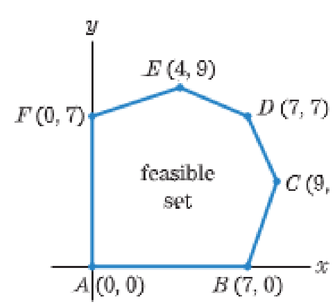


Figure 1

48. Let U be the set of vertices in Fig. 2. Let $R = \{\text{vertices } (x, y) \text{ with } x \geq 150\}$, $S = \{\text{vertices } (x, y) \text{ with } y \leq 100\}$, and $T = \{\text{vertices } (x, y) \text{ with } x + y \leq 400\}$. List the elements of the following sets.
(a) R (b) S (c) T
(d) $R \cap S'$ (e) $R' \cup T$ (f) $R' \cap S' \cap T'$

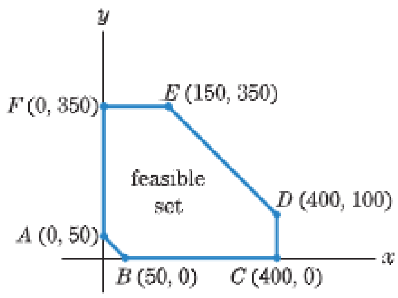


Figure 2

49. **Sandwich Toppings** Ed's Cheesesteaks offers any combination of three toppings on his sandwiches: peppers, onions, and mushrooms. How many different ways can you order a sandwich from Ed? List them.

50. **Toppings Choices** Amy ordered a baked potato at a restaurant. The server offered her butter, cheese, chives, and bacon as toppings. How many different ways could she have her potato? List them.

51. Let $S = \{1, 3, 5, 7\}$ and $T = \{2, 5, 7\}$. Give an example of a subset of T that is not a subset of S.

52. Suppose that S and T are subsets of the set U. Under what circumstance will $S \cap T = T$?

53. Suppose that S and T are subsets of the set U. Under what circumstance will $S \cup T = T$?

54. Find three subsets of the set of integers from 1 through 10, R, S, and T, such that $R \cup (S \cap T)$ is different from $(R \cup S) \cap T$.

In Exercises 55–62, determine whether the statement is true or false.

55. $5 \in \{3, 5, 7\}$
56. $\{1, 3\} \subseteq \{1, 2, 3\}$
57. $\{b\} \subseteq \{b, c\}$
58. $0 \in \{1, 2, 3\}$
59. $0 \in \varnothing$
60. $\varnothing \subseteq \{a, b, c\}$
61. $\{b, c\} \subseteq \{b, c\}$
62. $1 \notin \{1\}$

Solutions to Check Your Understanding 1

1. (a) $\{e, f, g\}$ (b) $\{c, d\}$
(c) $\{c\}$. This problem asks for the intersection of two sets. The first set is $R \cap S = \{c, d\}$, and the second set is $T = \{c, e, g\}$. The intersection of these sets is $\{c\}$.
(d) $\{c\}$. Here again, the problem asks for the intersection of two sets. However, now the first set is $R = \{a, b, c, d\}$ and the second set is $S \cap T = \{c, e\}$. The intersection of these sets is $\{c\}$.

Note: It should be expected that the set $(R \cap S) \cap T$ is the same as the set $R \cap (S \cap T)$, for each set consists of those elements that are in all three sets. Therefore, each of these sets equals the set $R \cap S \cap T$.

2. (a) $W' = \{\text{men who have received Nobel Prizes}\}$. This is so, since W' consists of those elements of U that are not in W—that is, those Nobel winners who are not women.
(b) $A \cap L' = \{\text{Americans who have received Nobel Prizes in fields other than literature}\}$
(c) $W \cap A \cap L' = \{\text{American women who have received Nobel Prizes in fields other than literature}\}$. This is so, since to qualify for $W \cap A \cap L'$, a Nobel winner must simultaneously be in W, in A, and in L'—that is, a woman, an American, and not a Nobel winner in literature.

3. $(A \cap W') \cup L$

2 A Fundamental Principle of Counting

A counting problem is one that requires us to determine the number of elements in a set S. Counting problems arise in many applications of mathematics and comprise the mathematical field of combinatorics. We shall study a number of different sorts of counting problems in the remainder of this chapter.

If S is any set, we will denote the number of elements in S by $n(S)$. For example, if $S = \{1, 7, 11\}$, then $n(S) = 3$, and if $S = \{a, b, c, d, e, f, g, h, i\}$, then $n(S) = 9$. Of course, if $S = \varnothing$, the empty set, then $n(S) = 0$.

Let us begin by stating one of the fundamental principles of counting, the **inclusion–exclusion principle**.

Inclusion–Exclusion Principle Let S and T be sets. Then,

$$n(S \cup T) = n(S) + n(T) - n(S \cap T). \tag{1}$$

Notice that formula (1) connects the four quantities $n(S \cup T)$, $n(S)$, $n(T)$, and $n(S \cap T)$. Given any three, the remaining quantity can be determined by using this formula.

To test the plausibility of the inclusion–exclusion principle, consider this example. Let $S = \{a, b, c, d, e\}$ and $T = \{a, c, g, h\}$. Then,

$$S \cup T = \{a, b, c, d, e, g, h\} \qquad n(S \cup T) = 7$$
$$S \cap T = \{a, c\} \qquad n(S \cap T) = 2.$$

In this case, the inclusion–exclusion principle reads

$$\begin{aligned} n(S \cup T) &= n(S) + n(T) - n(S \cap T) \\ 7 \quad &= \quad 5 \quad + \quad 4 \quad - \quad 2, \end{aligned}$$

which is correct.

Here is the reason for the validity of the inclusion–exclusion principle: The left side of formula (1) is $n(S \cup T)$, the number of elements in either S or T (or both). As a first approximation to this number, add the number of elements in S to the number of elements in T, obtaining $n(S) + n(T)$. However, if an element lies in both S and T, it is counted twice—once in $n(S)$ and again in $n(T)$. To make up for this double counting, we must subtract the number of elements counted twice, namely, $n(S \cap T)$. So doing gives us $n(S) + n(T) - n(S \cap T)$ as the number of elements in $S \cup T$.

When S and T are disjoint, the inclusion–exclusion principle reduces to a simple sum.

> **Special Case of the Inclusion–Exclusion Principle** If $S \cap T = \varnothing$, then
>
> $$n(S \cup T) = n(S) + n(T).$$

The next example illustrates a typical use of the inclusion–exclusion principle in an applied problem.

EXAMPLE 1 **Using the Inclusion–Exclusion Principle** In the year 2016, *Executive* magazine surveyed the presidents of the 500 largest corporations in the United States. Of these 500 people, 310 had degrees (of any sort) in business, 238 had undergraduate degrees in business, and 184 had graduate degrees in business. How many presidents had both undergraduate and graduate degrees in business?

SOLUTION Let

$$S = \{\text{presidents with an undergraduate degree in business}\}$$
$$T = \{\text{presidents with a graduate degree in business}\}.$$

Then,

$$S \cup T = \{\text{presidents with at least one degree in business}\}$$
$$S \cap T = \{\text{presidents with both undergraduate and graduate degrees in business}\}.$$

From the data given, we have

$$n(S) = 238 \qquad n(T) = 184 \qquad n(S \cup T) = 310.$$

The problem asks for $n(S \cap T)$. By the inclusion–exclusion principle, we have

$$\begin{aligned} n(S \cup T) &= n(S) + n(T) - n(S \cap T) \\ 310 &= 238 + 184 - n(S \cap T) \\ n(S \cap T) &= 112. \end{aligned}$$

That is, exactly 112 of the presidents had both undergraduate and graduate degrees in business.

» Now Try Exercise 9

Venn Diagrams

It is possible to visualize sets geometrically by means of drawings known as **Venn diagrams**. Such graphical representations of sets are very useful tools in solving counting problems. In order to describe Venn diagrams, let us begin with a single set S contained in a universal set U. Draw a rectangle, and view its points as the elements of U [Fig. 1(a)]. To show that S is a subset of U, we draw a circle inside the rectangle and view S as the set of points in the circle [Fig. 1(b)]. The resulting diagram is called a Venn diagram of S. It illustrates the proper relationship between S and U. Since S' consists of those elements of U that are not in S, we may view the portion of the rectangle that is outside of the circle as representing S' [Fig. 1(c)].

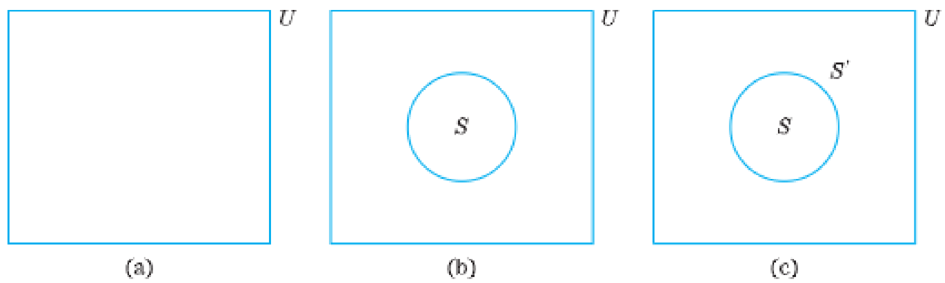


Figure 1

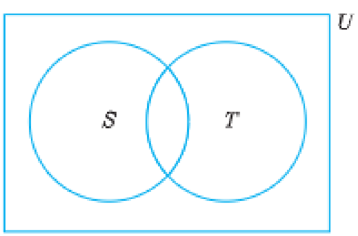


Figure 2

Venn diagrams are particularly useful for visualizing the relationship between two or three sets. Suppose that we are given two sets S and T in a universal set U. As before, we represent each of the sets by means of a circle inside the rectangle (Fig. 2).

We can now illustrate a number of sets by shading in appropriate regions of the rectangle. For instance, in Fig. 3(a), (b), and (c), we have shaded the regions corresponding to T, $S \cup T$, and $S \cap T$, respectively.

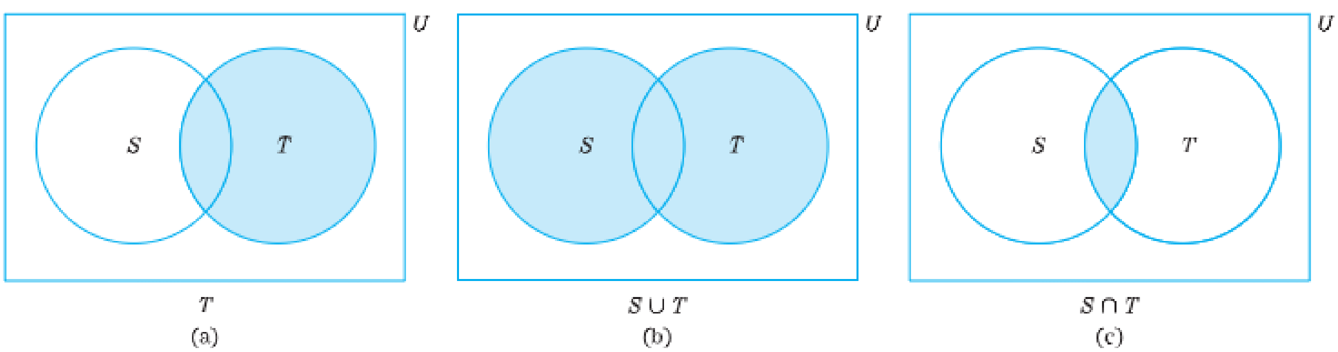


Figure 3

EXAMPLE 2 **Shading Portions of a Venn Diagram** Shade the portions of the rectangle corresponding to the sets

(a) $S \cap T'$ **(b)** $(S \cap T')'$.

SOLUTION

(a) $S \cap T'$ consists of the points in S and in T'—that is, the points in S and not in T [Figs. 4(a), 4(b) on the next page]. So we shade the points that are in the circle S but are not in the circle T [Fig. 4(c)].

(b) $(S \cap T')'$ is the complement of the set $S \cap T'$. Therefore, it consists of exactly those points not shaded in Fig. 4(c). [See Fig. 4(d).]

>> Now Try Exercises 15 and 19

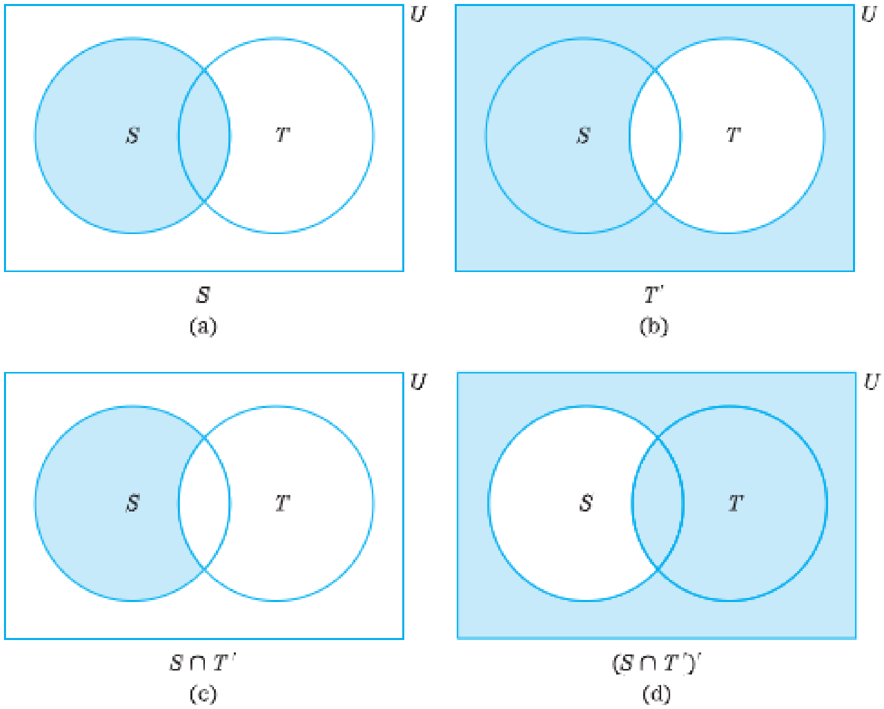


Figure 4

In a similar manner, Venn diagrams can illustrate intersections and unions of three sets. Some representative regions are shaded in Fig. 5.

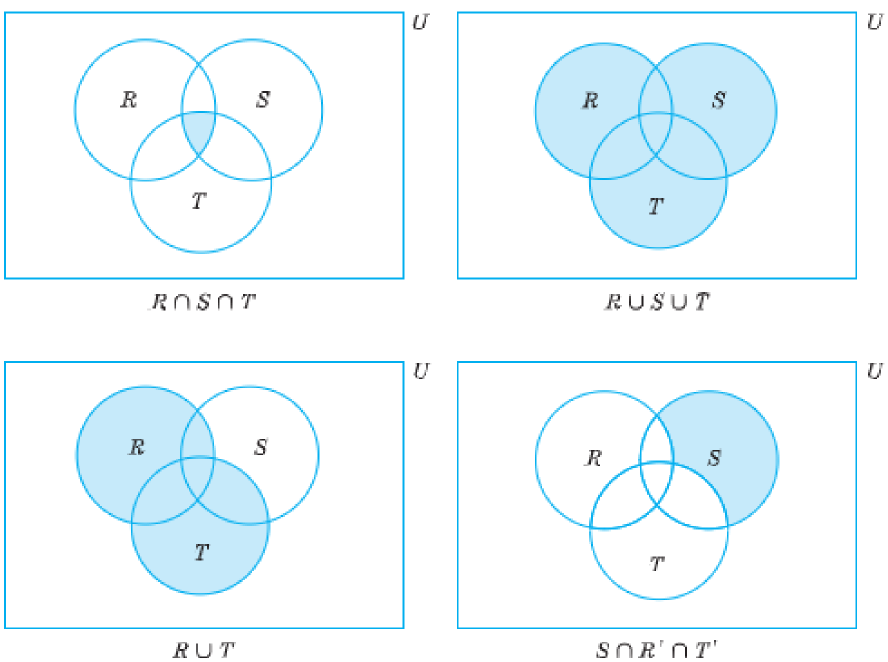


Figure 5

There are many formulas expressing relationships between intersections and unions of sets. Possibly the most fundamental are the two formulas known as *De Morgan's laws*.

De Morgan's Laws Let S and T be sets. Then,

$$(S \cup T)' = S' \cap T' \quad \text{and} \quad (S \cap T)' = S' \cup T'.$$

In other words, De Morgan's laws state that, to form the complement of a union (or intersection), form the complements of the individual sets and change unions to intersections (or intersections to unions).

Verification of De Morgan's Laws

Let us use Venn diagrams to describe $(S \cup T)'$. In Fig. 6(a), we have shaded the region corresponding to $S \cup T$. In Fig. 6(b), we have shaded the region corresponding to $(S \cup T)'$. In Figs. 6(c) and 4(d), we have shaded the regions corresponding to S' and T'. By considering the common shaded regions of Figs. 6(c) and (d), we arrive at the shaded region corresponding to $S' \cap T'$ [Fig. 6(e)]. Note that this is the same region as shaded in Fig. 6(b). Therefore,

$$(S \cup T)' = S' \cap T'.$$

This verifies the first of De Morgan's laws. The proof of the second law is similar.

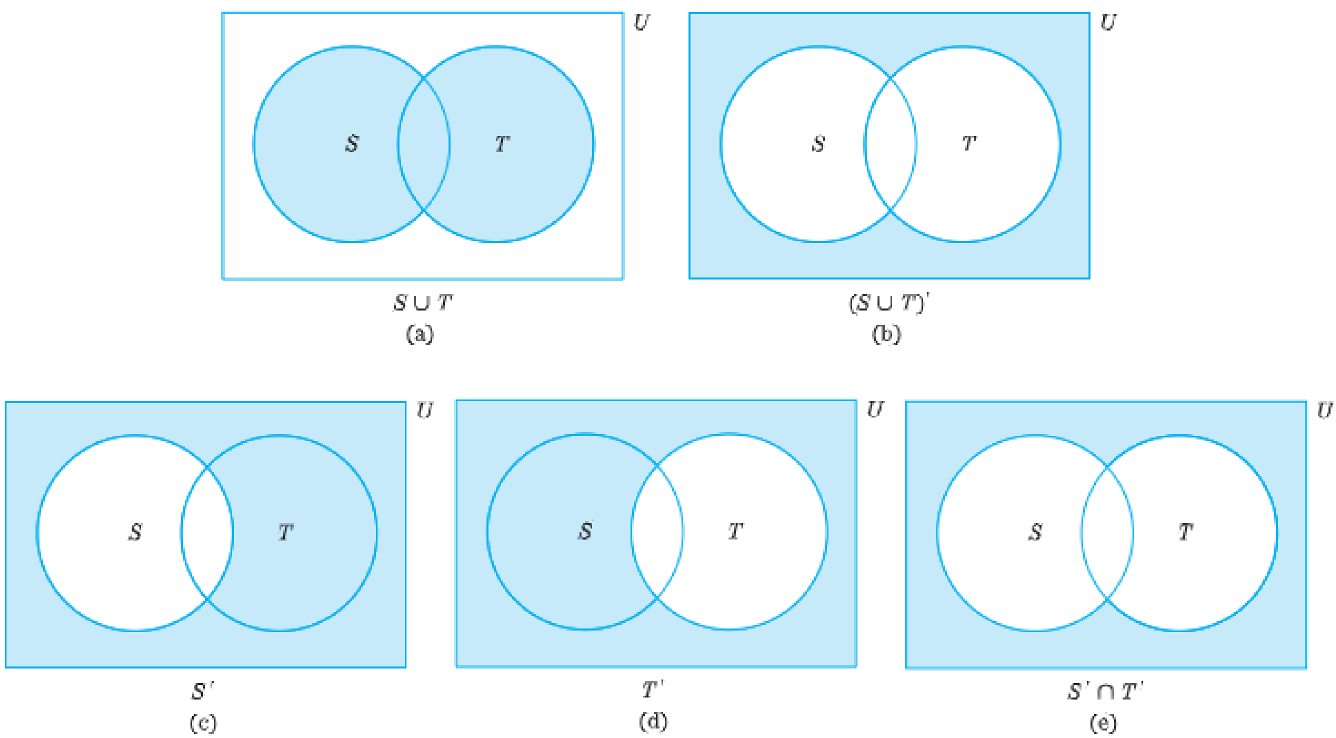


Figure 6

INCORPORATING TECHNOLOGY

WolframAlpha Venn diagrams can be displayed with instructions containing the words **intersect**, **union**, and **complement**. For instance, the instruction

S intersect (complement R) intersect (complement T)

produces the fourth Venn diagram of Fig. 5.

Check Your Understanding 2

Solutions can be found following the section exercises.

1. Draw a two-circle Venn diagram, and shade the portion corresponding to the set $(S \cap T') \cup (S \cap T)$.
2. What does the inclusion–exclusion principle conclude when T is a subset of S?

EXERCISES 2

1. Find $n(S \cup T)$, given that $n(S) = 4$, $n(T) = 4$, and $n(S \cap T) = 2$.
2. Find $n(S \cup T)$, given that $n(S) = 17$, $n(T) = 13$, and $n(S \cap T) = 0$.
3. Find $n(S \cap T)$, given that $n(S) = 6$, $n(T) = 9$, and $n(S \cup T) = 15$.
4. Find $n(S \cap T)$, given that $n(S) = 4$, $n(T) = 12$, and $n(S \cup T) = 15$.

5. Find $n(S)$, given that $n(T) = 7$, $n(S \cap T) = 5$, and $n(S \cup T) = 10$.

6. Find $n(T)$, given that $n(S) = 14$, $n(S \cap T) = 6$, and $n(S \cup T) = 14$.

7. If $n(S) = n(S \cap T)$, what can you conclude about S and T?

8. If $n(T) = n(S \cup T)$, what can you conclude about S and T?

9. **Languages** Suppose that each of the 314 million adults in South America is fluent in Portuguese or Spanish. If 170 million are fluent in Portuguese and 155 million are fluent in Spanish, how many are fluent in both languages?

10. **Course Enrollments** Suppose that all of the 1000 first-year students at a certain college are enrolled in a math or an English course. Suppose that 400 are taking both math and English and 600 are taking English. How many are taking a math course?

11. **Symmetry of Letters** Of the 26 capital letters of the alphabet, 11 have vertical symmetry (for instance, A, M, and T), 9 have horizontal symmetry (such as B, C, and D), and 4 have both (H, I, O, X). How many letters have neither horizontal nor vertical symmetry?

12. **Streaming Subscriptions** A survey of employees in a certain company revealed that 250 people subscribe to a streaming video service, 75 subscribe to a streaming music service, and 25 subscribe to both. How many people subscribe to at least one of these services?

13. **Automobile Options** Motors Inc. manufactured 325 cars with navigation systems, 216 with push-button start, and 89 with both of these options. How many cars were manufactured with at least one of the two options?

14. **Investments** A survey of 120 investors in stocks and bonds revealed that 90 investors owned stocks and 70 owned bonds. How many investors owned both stocks and bonds?

In Exercises 15–26, draw a two-circle Venn diagram and shade the portion corresponding to the set.

15. $S \cap T'$
16. $S' \cap T'$
17. $S' \cup T$
18. $S' \cup T'$
19. $(S \cap T')'$
20. $(S \cap T)'$
21. $(S \cap T') \cup (S' \cap T)$
22. $(S \cap T) \cup (S' \cap T')$
23. $S \cup (S \cap T)$
24. $S \cup (T' \cup S)$
25. $S \cup S'$
26. $S \cap S'$

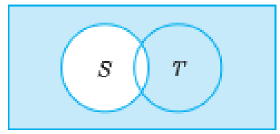
In Exercises 27–38, draw a three-circle Venn diagram and shade the portion corresponding to the set.

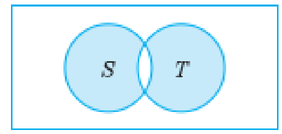
27. $R \cap S \cap T'$
28. $R' \cap S' \cap T$
29. $R \cup (S \cap T)$
30. $R \cap (S \cup T)$
31. $R \cap (S' \cup T)$
32. $R' \cup (S \cap T')$
33. $R \cap T$
34. $S \cap T'$
35. $R' \cup S' \cup T'$
36. $(R \cap S \cap T)'$
37. $(R \cap T) \cup (S \cap T')$
38. $(R \cup S') \cap (R \cup T')$

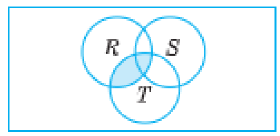
In Exercises 39–44, use De Morgan's laws to simplify each given expression.

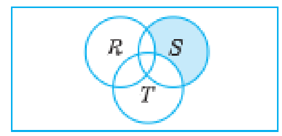
39. $S' \cup (S \cap T)'$
40. $S \cap (S \cup T)'$
41. $(S' \cup T)'$
42. $(S' \cap T')'$
43. $T \cup (S \cap T)'$
44. $(S' \cap T)' \cup S$

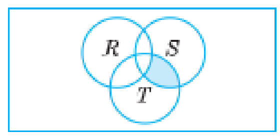
In Exercises 45–50, give a set-theoretic expression that describes the shaded portion of each Venn diagram.

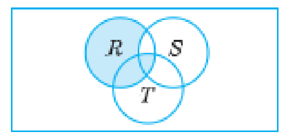
45.


46.


47.


48.


49.


50.


By drawing a Venn diagram, simplify each of the expressions in Exercises 51–54 to involve at most one union and the complement symbol applied only to R, S, and T.

51. $(T \cap S) \cup (T \cap R) \cup (R \cap S') \cup (T \cap R' \cap S')$

52. $(R \cap S) \cup (S \cap T) \cup (R \cap S' \cap T')$

53. $((R \cap S') \cup (S \cap T') \cup (T \cap R'))'$

54. $(R \cap T) \cup (R \cap S) \cup (S \cap T') \cup (R \cap S' \cap T')$

Citizenship Assume that the universal set U is the set of all people living in the United States. Let A be the set of all U.S. citizens, let B be the set of all children under 5 years of age, let C be the set of children from 5 to 18 years of age, let D be the set of everyone over the age of 18, and let E be the set of all people who are employed. Describe in words each set in Exercises 55–60.

55. $A' \cup (D \cap E)$
56. $A \cap C \cap E$
57. $D \cap E'$
58. $A \cap (D \cup E)$
59. $A' \cap B'$
60. $B \cap E$

Solutions to Check Your Understanding 2

1. $(S \cap T') \cup (S \cap T)$ is given as a union of two sets, $S \cap T'$ and $S \cap T$. The Venn diagrams for these two sets are given in Figs. 7(a) and (b). The desired set consists of the elements that are in one or the other (or both) of the two sets. Therefore, its Venn diagram is obtained by shading everything that is shaded in either Fig. 7(a) or (b). [See Fig. 7(c).] *Note:* Looking at Fig. 7(c) reveals that $(S \cap T') \cup (S \cap T)$ and S are the same set. Often, Venn diagrams can be used to simplify complicated set-theoretic expressions.

2. When $T \subseteq S$, $S \cup T = S$ *and* $S \cap T = T$; the inclusion–exclusion principle becomes

$$n(S \cup T) = n(S) + n(T) - n(S \cap T)$$
$$n(S) = n(S) + n(T) - n(T)$$
$$n(S) = n(S).$$

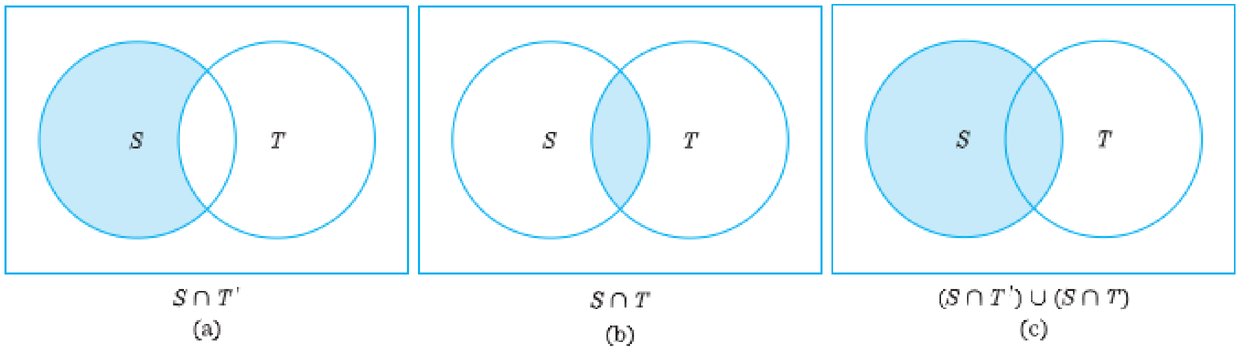


Figure 7

3 Venn Diagrams and Counting

In this section, we discuss the use of Venn diagrams in solving counting problems. The techniques developed are especially useful in analyzing survey data.

Each Venn diagram divides the universal set U into a certain number of regions. For example, the Venn diagram for a single set divides U into two regions—the inside and outside of the circle [Fig. 1(a)]. The Venn diagram for two sets divides U into four regions [Fig. 1(b)]. The Venn diagram for three sets divides U into eight regions [Fig. 1(c)]. Each of the regions is called a **basic region** for the Venn diagram. Knowing the number of elements in each basic region is of great use in many applied problems. As an illustration, consider the next example.

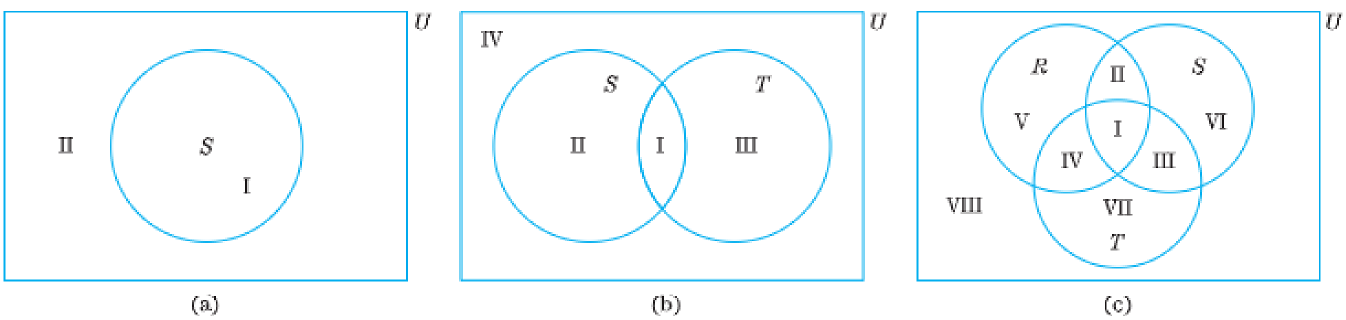


Figure 1

EXAMPLE 1 **Nobel Prize Winners** Let

$$U = \{\text{Nobel winners during the period 1901–2015}\}$$
$$A = \{\text{American Nobel winners during the period 1901–2015}\}$$
$$C = \{\text{Chemistry Nobel winners during the period 1901–2015}\}$$
$$P = \{\text{Nobel Peace Prize winners during the period 1901–2015}\}.$$

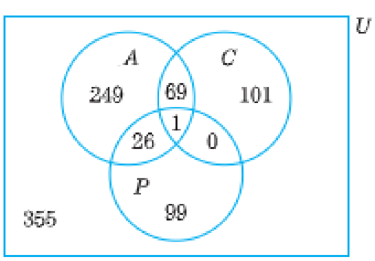


Figure 2

These sets are illustrated in the Venn diagram of Fig. 2, in which each basic region has been labeled with the number of elements in it.

(a) How many Americans received a Nobel Prize during this period 1901–2015?

(b) How many Americans received Nobel Prizes in fields other than chemistry and peace during this period?

(c) How many Americans received the Nobel Peace Prize during this period?

(d) How many Nobel Prize winners were there during this period?

SOLUTION **(a)** The number of Americans who received a Nobel Prize is the total contained in the circle A [Fig. 3(a) on the next page], which is

$$249 + 26 + 1 + 69 = 345.$$

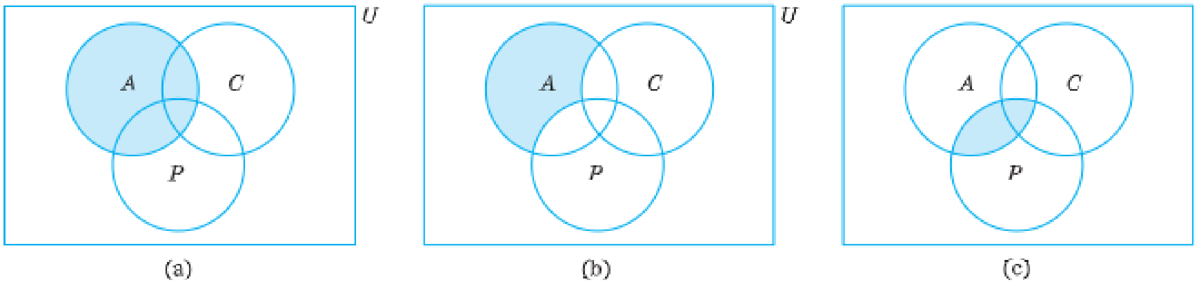


Figure 3

(b) The question asks for the number of Nobel winners in A but not in C and not in P. So start with the A circle, and eliminate those basic regions belonging to C or P [Fig. 3(b)]. There remains a single basic region with 249. Nobel winners. Note that this region corresponds to $A \cap C' \cap P'$.

(c) The question asks for the number of elements in both A and P—that is, $n(A \cap P)$. But $A \cap P$ comprises two basic regions [Fig. 3(c)]. Thus, to compute $n(A \cap P)$, we add the numbers in these basic regions to obtain $26 + 1 = 27$ Americans who have received the Nobel Peace Prize.

(d) The number of recipients is just $n(U)$, and we obtain it by adding together the numbers corresponding to the basic regions. We obtain

$$355 + 249 + 69 + 1 + 26 + 101 + 0 + 99 = 900.$$

>> Now Try Exercises 1–4

One need not always be given the number of elements in each of the basic regions of a Venn diagram. Very often, this data can be deduced from given information.

EXAMPLE 2 **Corporate Presidents** Consider the set of 500 corporate presidents of Example 1, Section 2.

(a) Draw a Venn diagram displaying the given data, and determine the number of elements in each basic region.

(b) Determine the number of presidents having exactly one degree (graduate or undergraduate) in business.

SOLUTION **(a)** Recall that we defined the following sets:

$$S = \{\text{presidents with an undergraduate degree in business}\}.$$
$$T = \{\text{presidents with a graduate degree in business}\}.$$

We were given the following data:

$$n(U) = 500 \qquad n(S) = 238 \qquad n(T) = 184 \qquad n(S \cup T) = 310.$$

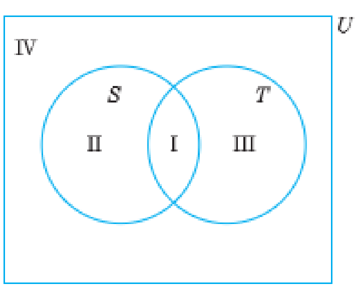


Figure 4

We draw a Venn diagram corresponding to S and T (Fig. 4). Notice that none of the given information corresponds to a basic region of the Venn diagram. So we must use our wits to determine the number of presidents in each of the regions I–IV. Region IV is the complement of $S \cup T$, so it contains

$$n(U) - n(S \cup T) = 500 - 310 = 190$$

presidents. Region I is just $S \cap T$. By using the inclusion–exclusion principle, in Example 1, Section 2, we determined that $n(S \cap T) = 112$. Now, the total number of presidents in I and II combined equals $n(S)$, or 238. Therefore, the number of presidents in II is

$$n(S) - n(S \cap T) = 238 - 112 = 126.$$

Similarly, the number of presidents in III is

$$n(T) - n(S \cap T) = 184 - 112 = 72.$$

Thus, we may fill in the data to obtain a completed Venn diagram (Fig. 5).

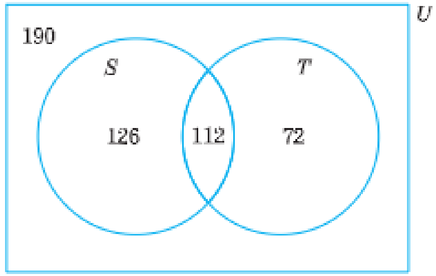


Figure 5

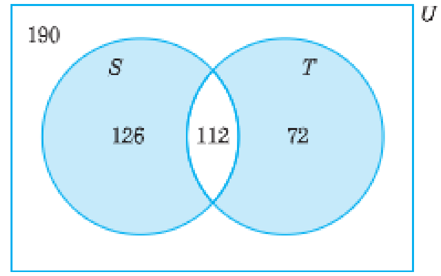


Figure 6

(b) The number of people with exactly one business degree corresponds to the shaded region in Fig. 6. Adding together the number of presidents in each of these regions gives $126 + 72 = 198$ presidents with exactly one business degree.

>> Now Try Exercise 23

Here is another example illustrating the procedure for determining the number of elements in each of the basic regions of a Venn diagram.

EXAMPLE 3 **Advertising Media** An advertising agency finds that the media use of its 170 clients is as follows:

115 use television (T)
100 use the Internet (I)
130 use mobile apps (M)
75 use television and the Internet
95 use the Internet and mobile apps
85 use television and mobile apps
70 use all three.

Use this data to complete the Venn diagram in Fig. 7 to display the clients' use of mass media.

SOLUTION Of the various data given, only the last item corresponds to one of the eight basic regions of the Venn diagram—namely, the "70" corresponding to the use of all three media. So we begin by entering this number in the diagram [Fig. 8(a)]. We can fill in the rest of the Venn diagram by working with the remaining information one piece at a time in the reverse order that it is given. Since 85 clients advertise in television and mobile apps, $85 - 70 = 15$ advertise in television and mobile apps but not on the Internet. The appropriate region is labeled in Fig. 8(b). In Fig. 8(c), the next two pieces of information have been used in the same way to fill in two more basic regions. In Fig. 8(c), we observe that three of the four basic regions comprising M have been filled in. Since $n(M) = 130$, we deduce that the number of clients advertising only in

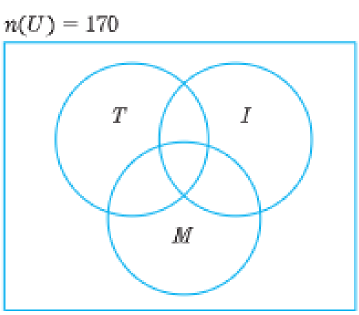


Figure 7

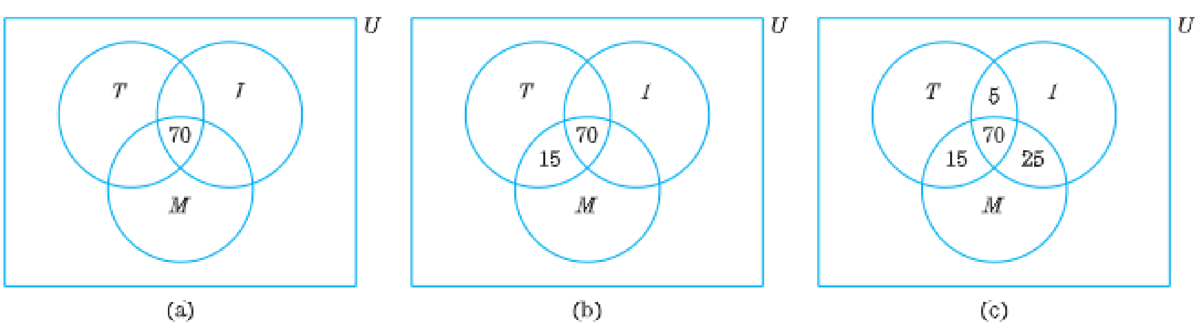


Figure 8

mobile apps is $130 - (15 + 70 + 25) = 130 - 110 = 20$ [Fig. 9(a)]. By similar reasoning, the number of clients using only Internet advertising and the number using only television advertising can be determined [Fig. 9(b)]. Adding together the numbers in the three circles gives the number of clients utilizing television, Internet, or mobile apps as $25 + 5 + 0 + 15 + 70 + 25 + 20 = 160$. Since there were 170 clients in total, the remainder—or $170 - 160 = 10$ clients—use none of these media. Figure 9(c) gives a complete display of the data.

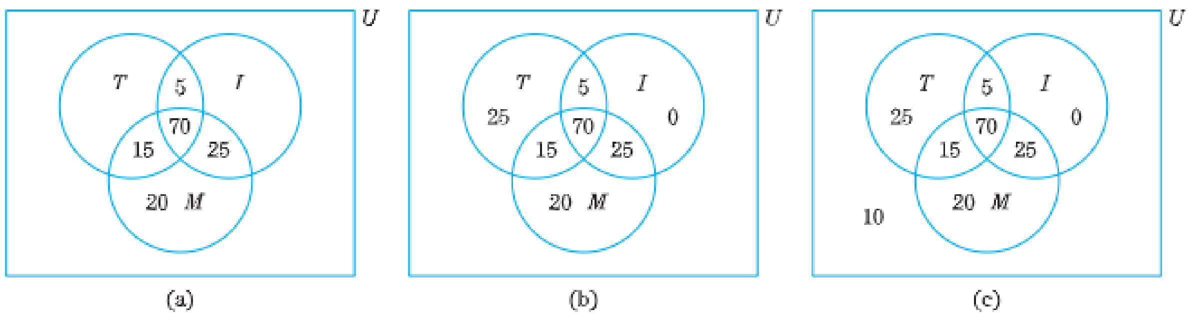


Figure 9

» Now Try Exercise 31(a)

Check Your Understanding 3

Solutions can be found following the section exercises.

1. Of the 1000 first-year students at a certain college, 700 take mathematics courses, 300 take mathematics and economics courses, and 200 do not take any mathematics or economics courses. Represent this data in a Venn diagram.

2. Refer to the Venn diagram from Problem 1.
   (a) How many of the first-year students take an economics course?
   (b) How many take an economics course but not a mathematics course?

EXERCISES 3

Family Library The Venn diagram in Fig. 10 classifies the 100 books in a family's library as hardback (H), fiction (F), and children's (C). Exercises 1–10 refer to this Venn diagram.

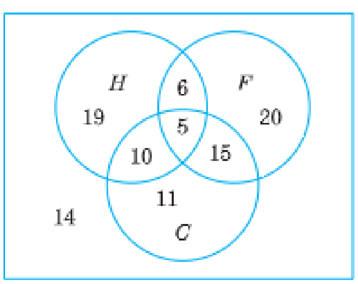


Figure 10

1. How many books are hardback fiction?
2. How many books are paperback fiction?
3. How many books are fiction?
4. How many books are nonfiction?
5. How many books are paperback nonfiction children's books?
6. How many books are adult hardback nonfiction?
7. How many books are either hardback or fiction?
8. How many hardback books are either fiction or children's books?
9. How many children's books are either hardback or fiction?
10. How many books are either hardback, fiction, or children's books?

In Exercises 11–22, let R, S, and T be subsets of the universal set U. Draw an appropriate Venn diagram, and use the given data to determine the number of elements in each basic region.

11. $n(U) = 17, n(S) = 12, n(T) = 7, n(S \cap T) = 5$
12. $n(U) = 20, n(S) = 11, n(T) = 7, n(S \cap T) = 7$
13. $n(U) = 20, n(S) = 12, n(T) = 14, n(S \cup T) = 18$
14. $n(S') = 4, n(S \cup T) = 12, n(S \cap T) = 5, n(T) = 9$
15. $n(U) = 75, n(S) = 15, n(T) = 25, n(S' \cap T') = 40$
16. $n(S) = 10, n(T) = 10, n(S \cap T) = 5, n(S') = 13$
17. $n(S) = 3, n(S \cup T) = 6, n(T) = 4, n(S' \cup T') = 9$
18. $n(U) = 15, n(S) = 8, n(T) = 9, n(S \cup T) = 14$
19. $n(U) = 28, n(R) = 12, n(S) = 12, n(T) = 9, n(R \cap S) = 5, n(S \cap T) = 3, n(R \cap T) = 7, n(R \cap S \cap T) = 2$
20. $n(U) = 29, n(R) = 10, n(S) = 12, n(T) = 10, n(R \cap S) = 1, n(R \cap T) = 5, n(S \cap T) = 4, n(R \cap S \cap T) = 1$
21. $n(R') = 22, n(R \cup S) = 21, n(S) = 14, n(T) = 22, n(R \cap S) = 7, n(S \cap T) = 9, n(R \cap T) = 11, n(R \cap S \cap T) = 5$
22. $n(U) = 64, n(R \cup S \cup T) = 45, n(R) = 22, n(T) = 26, n(R \cap S) = 4, n(S \cap T) = 6, n(R \cap T) = 8, n(R \cap S \cap T) = 1$
23. **Music Preferences** A survey of 70 high school students revealed that 35 like rock music, 15 like hip-hop music, and 5 like both. How many of the students surveyed do not like either rock or hip-hop music?

24. **Nobel Winners** A total of 900 Nobel Prizes had been awarded by 2015. Fourteen of the 112 prizes in literature were awarded to Scandinavians. Scandinavians received a total of 57 awards. How many Nobel Prizes outside of literature have been awarded to non-Scandinavians?

25. **Analysis of Sonnet** One of Shakespeare's sonnets has a verb in 11 of its 14 lines, an adjective in 9 lines, and both in 7 lines. How many lines have a verb but no adjective? An adjective but no verb? Neither an adjective nor a verb?

Exam Performance The results from an exam taken by 150 students were as follows:

90 students correctly answered the first question,
71 students correctly answered the second question,
66 students correctly answered both questions.

Exercises 26–30 refer to these students.

26. How many students correctly answered either the first or second question?

27. How many students did not answer either of the two questions correctly?

28. How many students answered either the first or the second question correctly, but not both?

29. How many students answered the second question correctly, but not the first?

30. How many students missed the second question?

31. **Class Enrollment** Out of 35 students in a finite math class, 22 are male, 19 are business majors, 27 are first-year students, 14 are male business majors, 17 are male first-year students, 15 are first-year business majors, and 11 are male first-year business majors.
    (a) Use this data to complete a Venn diagram displaying the characteristics of the students.
    (b) How many students in the class are neither first-year, nor male, nor business majors?
    (c) How many non-male business majors are in the class?

32. **Exercise Preferences** A survey of 100 college faculty who exercise regularly found that 45 jog, 30 swim, 20 cycle, 6 jog and swim, 1 jogs and cycles, 5 swim and cycle, and 1 does all three. How many of the faculty members do not do any of these activities? How many just jog?

Flag Colors The three most common colors in the 193 flags of the member nations of the United Nations are red, white, and blue:

52 flags contain all three colors
103 flags contain both red and white
66 flags contain both red and blue
73 flags contain both white and blue
145 flags contain red
132 flags contain white
104 flags contain blue.

Exercises 33–38 refer to the 193 flags.

33. How many flags contain red, but not white or blue?

34. How many flags contain exactly one of the three colors?

35. How many flags contain none of the three colors?

36. How many flags contain exactly two of the three colors?

37. How many flags contain red and white, but not blue?

38. How many flags contain red or white, but not both?

News Dissemination A merchant surveyed 400 people to determine from what source they found out about an upcoming sale. The results of the survey follow:

180 from the Internet
190 from television
190 from newspapers
80 from the Internet and television
90 from the Internet and newspapers
50 from television and newspapers
30 from all three sources.

Exercises 39–44 refer to the people in this survey.

39. How many people learned of the sale from newspapers or the Internet but not from both?

40. How many people learned of the sale only from newspapers?

41. How many people learned of the sale from the Internet or television but not from newspapers?

42. How many people learned of the sale from at least two of the three media?

43. How many people learned of the sale from exactly one of the three media?

44. How many people learned of the sale from the Internet and television but not from newspapers?

45. **Course Enrollments** Table 1 shows the number of students enrolled in each of three science courses at Gotham College. Although no students are enrolled in all three courses, 15 are enrolled in both chemistry and physics, 10 are enrolled in both physics and biology, and 5 are enrolled in both biology and chemistry. How many students are enrolled in at least one of these science courses?

Table 1

Course	Enrollment
Chemistry	60
Physics	40
Biology	30

Foreign Language Courses A survey in a local high school shows that, of the 4000 students in the school,

2000 take French (F)
3000 take Spanish (S)
500 take Latin (L)
1500 take both French and Spanish
300 take both French and Latin
200 take Spanish and Latin
50 take all three languages.

Use a Venn diagram to find the number of people in the sets given in Exercises 46–50.

46. $L \cap (F \cup S)$
47. $(L \cup F \cup S)'$
48. L'
49. $L \cup S \cup F$
50. $F \cap S' \cap L'$

51. **Voting Preferences** One hundred college students were surveyed after voting in an election involving a Democrat and a Republican. There were 50 first-year students, 55 voted Democratic, and 25 were non-first-year students who voted Republican. How many first-year students voted Democratic?

52. **Union Membership and Education Status** A group of 100 workers were asked whether they were college graduates and whether they belonged to a union. According to their responses, 60 were not college graduates, 20 were nonunion college graduates, and 30 were union members. How many of the workers were neither college graduates nor union members?

53. **Diagnostic Test Results** A class of 30 students was given a diagnostic test on the first day of a mathematics course. At the end of the semester, only 2 of the 21 students who had passed the diagnostic test failed the course. A total of 23 students passed the course. How many students managed to pass the course even though they failed the diagnostic test?

54. **Air-Traffic Controllers** A group of applicants for training as air-traffic controllers consists of 35 pilots, 20 veterans, 30 pilots who were not veterans, and 50 people who were neither veterans nor pilots. How large was the group?

College Majors A group of 61 students has the following characteristics:

6 are biology majors and seniors
17 are biology majors and not seniors
12 are not seniors and are majoring in a field other than biology.

Exercises 55–60 refer to these students.

55. How many of the students are either seniors or biology majors?

56. How many of the students are seniors?

57. How many of the students are not seniors?

58. How many of the students are biology majors?

59. How many of the seniors are not biology majors?

60. How many of the students are not biology majors?

Music Preferences A campus radio station surveyed 190 students to determine the genres of music they liked. The survey results follow:

114 like rock
50 like country
15 like rock and rap
11 like rap and country
20 like rap only
10 like rock and rap, but not country
9 like rock and country, but not rap
20 don't like any of the three types of music.

Exercises 61–68 refer to the students in this survey.

61. How many students like rock only?

62. How many students like country but not rock?

63. How many students like rap and country but not rock?

64. How many students like rap or country but not rock?

65. How many students like exactly one of the genres?

66. How many students like all three genres?

67. How many students like at least two of the three genres?

68. How many students do not like either rock or country?

69. **Website Preferences** One hundred and sixty business executives were surveyed to determine whether they regularly visit the *CNN Money*, *Bloomberg*, or *The Wall Street Journal* websites. The survey showed that 70 visit *CNN Money*, 60 visit *Bloomberg*, 55 visit *The Wall Street Journal*, 45 visit exactly two of the three websites, 20 visit *CNN Money* and *Bloomberg*, 20 visit *Bloomberg* and *The Wall Street Journal*, and 5 visit all three websites. How many do not visit any of the three websites?

70. **Small Businesses** A survey of the characteristics of 100 small businesses that had failed revealed that 95 of them either were undercapitalized, had inexperienced management, or had a poor location. Four of the businesses had all three of these characteristics. Forty businesses were undercapitalized but had experienced management and good location. Fifteen businesses had inexperienced management but sufficient capitalization and good location. Seven were undercapitalized and had inexperienced management. Nine were undercapitalized and had poor location. Ten had inexperienced management and poor location. How many of the businesses had poor location? Which of the three characteristics was most prevalent in the failed businesses?

71. **Music** Each of the 100 students attending a conservatory of music plays at least one of three instruments: piano, violin, and clarinet. Of the students, 65 play the piano, 42 play the violin, 28 play the clarinet, 20 play the piano and the violin, 10 play the violin and the clarinet, and 8 play the piano and the clarinet. How many play all three instruments? *Hint:* Let x represent the number of students who play all three instruments.

72. **Courses** Students living in a certain dormitory were asked about their enrollment in mathematics and history courses. Ten percent were taking both types of courses, and twenty percent were taking neither type of course. One hundred sixty students were taking a mathematics course but not a history course, and one hundred twenty students were taking a history course but not a mathematics course. How many students were taking a mathematics course? *Hint:* Let x be the total number of students living in the dormitory.

Solutions to Check Your Understanding 3

1. Draw a Venn diagram with two circles, one for mathematics (M) and one for economics (E) [Fig. 11(a)]. This Venn diagram has four basic regions, and our goal is to label each basic region with the proper number of students. The numbers for two of the basic regions are given directly. Since "300 take mathematics and economics," $n(M \cap E) = 300$. Since "200 do not take any mathematics or economics courses," $n((M \cup E)') = 200$ [Fig. 11(b)]. Now, "700 take mathematics courses." Since M is made up of two basic regions and one region has 300 elements, the other basic region of M must contain 400 elements [Fig. 11(c)]. At this point, all but one of the basic regions have been labeled and $400 + 300 + 200 = 900$ students have been accounted for. Since there is a total of 1000 students, the remaining basic region has 100 students [Fig. 11(d)].

2. (a) 400. "Economics" refers to the entire circle E, which is made up of two basic regions, one having 300 elements and the other 100. (A common error is to interpret the question as asking for the number of first-year students who take economics exclusively and therefore give the answer 100. To say that a person takes an economics course does not imply anything about the person's enrollment in mathematics courses.)

   (b) 100

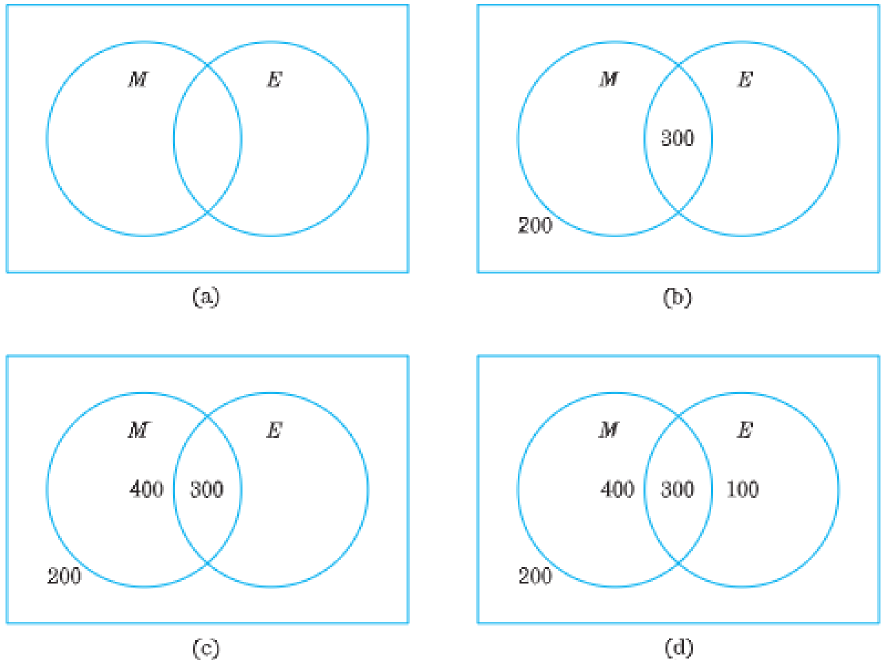


Figure 11

4 The Multiplication Principle

In this section, we introduce a second fundamental principle of counting, the *multiplication principle*. By way of motivation, consider the following example:

EXAMPLE 1 **Counting Paths through a Maze** A medical researcher wishes to test the effect of a drug on a rat's perception by studying the rat's ability to run a maze while under the influence of the drug. The maze is constructed so that, to arrive at the exit point C, the rat must pass through a central point B. There are five paths from the entry point A to B, and three paths from B to C. In how many different ways can the rat run the maze from A to C? (See Fig. 1.)

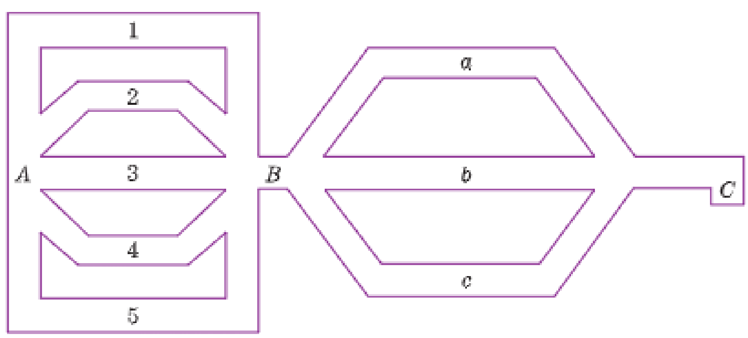


Figure 1

SOLUTION The paths from A to B have been labeled 1 through 5, and the paths from B to C have been labeled a through c. The various paths through the maze can be schematically represented as in Fig. 2 on the next page. The diagram shows that there are five ways to go from A to B. For each of these five ways, there are three ways to go from B to C. So there are five groups of three paths each and therefore $5 \cdot 3 = 15$ possible paths from A to C. (A diagram such as Fig. 2, called a **tree diagram**, is useful in enumerating the various possibilities in counting problems.)

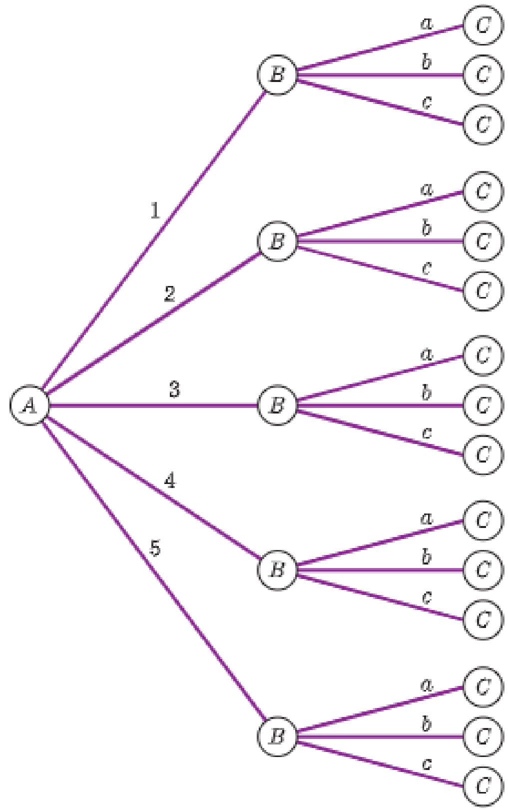


Figure 2

» *Now Try Exercise 1*

In the preceding problem, selecting a path is a task that can be broken up into two consecutive choices.

Choose path from A to B	Choose path from B to C
Choice 1	Choice 2

The first choice can be performed in five ways, and after the first choice has been carried out, the second can be performed in three ways. And we determined that the entire task can be performed in $5 \cdot 3 = 15$ ways. The same reasoning as just used yields the following useful counting principle.

> **Multiplication Principle** Suppose that a task is composed of two consecutive choices. If choice 1 can be performed in m ways and, for each of these, choice 2 can be performed in n ways, then the complete task can be performed in $m \cdot n$ ways.

EXAMPLE 2 **Counting Routes for a Trip** An airline passenger must fly from New York to Frankfurt via London. There are 8 flights leaving New York for London. All of these provide connections on any one of 19 flights from London to Frankfurt. In how many different ways can the passenger book reservations?

SOLUTION The task *Fly from New York to Frankfurt* is composed of two consecutive choices:

Select a flight from New York to London	Select a flight from London to Frankfurt
Choice 1	Choice 2

From the data given, the multiplication principle implies that the task can be accomplished in $8 \cdot 19 = 152$ ways.

» *Now Try Exercise 3*

It is possible to generalize the multiplication principle to tasks consisting of more than two choices.

Generalized Multiplication Principle Suppose that a task consists of t choices performed consecutively. Suppose that choice 1 can be performed in m_1 ways; for each of these, choice 2 in m_2 ways; for each of these, choice 3 in m_3 ways; and so forth. Then the task can be performed in

$$m_1 \cdot m_2 \cdot m_3 \cdot \cdots \cdot m_t \quad \text{ways.}$$

EXAMPLE 3 **Officers for a Board of Directors** A corporation has a board of directors consisting of 10 members. The board must select from among its members a chairperson, vice chairperson, and secretary. In how many ways can this be done?

SOLUTION The task *Select the three officers* can be divided into three consecutive choices:

Select chairperson	Select vice chairperson	Select secretary

Since there are 10 directors, choice 1 can be performed in 10 ways. After the chairperson has been selected, there are 9 directors left as possible candidates for vice chairperson so that for each way of performing choice 1, choice 2 can be performed in 9 ways. After this has been done, there are 8 directors who are possible candidates for secretary, so choice 3 can be performed in 8 ways. By the generalized multiplication principle, the number of possible ways to perform the sequence of three choices equals $10 \cdot 9 \cdot 8$, or 720. So the officers of the board can be selected in 720 ways.

>> Now Try Exercise 7

In Example 3, we made important use of the phrase "for each of these" in the generalized multiplication principle. The choice *Select a vice chairperson* can be performed in 10 ways, since any member of the board is eligible. However, when we view the selection process as a sequence of choices of which *Select a vice chairperson* is the second choice, the situation has changed. *For each way* that the first choice is performed, one person will have been used up; hence, there will be only 9 possibilities for choosing the vice chairperson.

Note that the order of the choices doesn't matter. For example, we could choose the vice chairman first, then the secretary, then the chairperson, and we would arrive at the same result.

EXAMPLE 4 **Posing for a Group Picture** In how many ways can a baseball team of nine players arrange themselves in a line for a group picture?

SOLUTION Choose the players by their place in the picture—say, from left to right. The first can be chosen in nine ways; for each of these outcomes, the second can be chosen in eight ways; for each of these outcomes, the third can be chosen in seven ways; and so forth. So the number of possible arrangements is

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880.$$

>> Now Try Exercise 11

We can write the product $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ from the previous example in a condensed way by using **factorial** notation:

DEFINITION Factorial If n is a positive integer, the number n **factorial**, denoted $n!$, is defined to be the product

$$n! = n(n-1)(n-2)\cdots(2)(1).$$

In addition, we define $0! = 1$.

EXAMPLE 5 **License Plates** A certain state uses automobile license plates that consist of three letters followed by three digits. How many such license plates are there?

SOLUTION The task in this case, *Form a license plate*, consists of a sequence of six choices: three for choosing letters and three for choosing digits. Each letter can be chosen in 26 ways and each digit in 10 ways. So the number of license plates is

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000.$$

>> Now Try Exercise 17

Check Your Understanding 4

Solutions can be found following the section exercises.

1. There are five seats available in a sedan. In how many ways can five people be seated if only three can drive?
2. A multiple-choice exam contains 10 questions, each having 3 possible answers. Assuming you answer each question, how many different ways are there of completing the exam?

EXERCISES 4

1. Jolene wants to drive from her house to the grocery store and then to the library. If her GPS suggests four routes from her house to the grocery store, and two routes from the grocery store to the library, how many total ways are there for Jolene to do this? (See Fig. 3.)

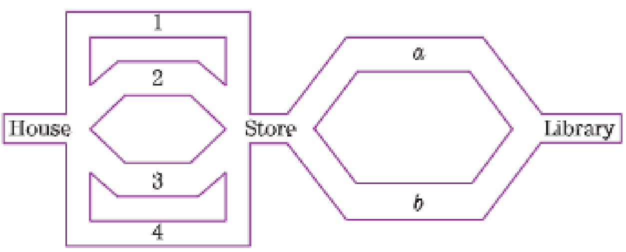


Figure 3

2. There are three bridges from the west shore of a river to an island in the center, and three bridges from the island to the east shore. How many different ways are there to cross from the west shore to the east shore? (See Fig. 4.)

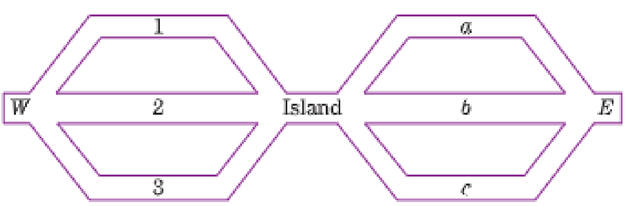


Figure 4

3. **Travel Options** If you can travel from Frederick, Maryland, to Baltimore, Maryland, by car, bus, or train and from Baltimore to London by airplane or ship, how many different ways are there to go from Frederick to London via Baltimore?
4. **Travel Options** Suppose that Maria wants to go from Florida to Maine via New York and can travel each leg of the journey by bus, car, train, or airplane. How many different ways can Maria make the trip?
5. **Daytona 500** Forty-four race cars competed in the 2016 Daytona 500. Assuming no ties, how many possibilities were there for the first-, second-, and third-place finishers?
6. **Kentucky Derby** Twenty horses competed in the 2016 Kentucky Derby. Assuming no ties, how many possibilities were there for the first, second, and third place finishers?
7. **Winners** Twenty athletes enter an Olympic event. Assuming no ties, how many different possibilities are there for winning the Gold Medal, Silver Medal, and Bronze Medal?
8. **Ranking Teams** A sportswriter is asked to rank six teams. How many different orderings are possible?
9. **Electing Captains** A football squad can elect a captain and an assistant captain in 870 possible ways. How many members does the squad have?
10. **Club Officers** A club can elect a president and a treasurer in 600 different ways. How many members does the club have?
11. **Group Picture** A group of five boys and three girls is to be photographed.
    (a) How many ways can they be arranged in one row?
    (b) How many ways can they be arranged with the girls in the front row and the boys in the back row?
12. **Arranging Books** Three history books and six novels are to be arranged on a bookshelf.
    (a) How many ways can they be arranged?
    (b) How many ways can they be arranged with the history books to the left of the novels?
13. **Rearranging Letters** How many different words (including nonsense words) can be formed by using the four letters of the word "MATH"?
14. **Three-Letter Words** How many different three-letter words (including nonsense words) are there in which successive letters are different?
15. **Selecting an Outfit** How many different outfits consisting of a coat and a hat can be selected from two coats and three hats?
16. **Selecting an Outfit** How many different outfits can be selected from two coats, four hats, and two scarves?
17. **Serial Numbers** A computer manufacturer assigns serial numbers to its computers. The first symbol of a serial number is either A, B, or C, indicating the manufacturing plant. The second and third symbols taken together are one of the numbers 01, 02, . . . , 12, indicating the month of manufacture. The final four symbols are digits. How many possible serial numbers are there?
18. **License Plates** Suppose that a license plate consists of a nonzero digit followed by three letters and then three nonzero digits. How many such license plates are there?

19. **Social Security Numbers** How many Social Security numbers are available if the only restriction is that the number 000-00-0000 cannot be assigned?

20. **Call Letters** In 1923, the Federal Communications Commission directed that all new radio stations east of the Mississippi River have call letters beginning with the letter W. How many different three- or four-letter call letters are possible?

21. **Area Codes** Before 1995, three-digit area codes for the United States had the following restrictions:
   (i) Neither 0 nor 1 could be used as the first digit.
   (ii) 0 or 1 had to be used for the second digit.
   (iii) There were no restrictions on the third digit.
   How many different area codes were possible?

22. **Area Codes** Refer to Exercise 21. Beginning in 1995, restriction (ii) was lifted and any digit could be used in the second position. How many different area codes were then possible?

Palindromes A number or word is said to be a *palindrome* if it reads the same backward as forward (e.g., 58485 or radar).

23. How many 5-digit numbers are palindromes?

24. How many 6-digit numbers are palindromes?

25. How many 4-letter words (including nonsense words) are palindromes?

26. How many 3-letter words (including nonsense words) are palindromes?

27. **World Series** The World Series of Baseball is played between the American League and National League champions, in which each league consists of 15 teams. How many different possible matchups are there for the World Series?

28. **Super Bowl** The Super Bowl is a game played between the National Football Conference and American Football Conference champions. Each conference consists of 16 teams. How many different possible matchups are there for the Super Bowl?

29. **Bridge Games** There are 3200 duplicate bridge clubs sanctioned by the American Contract Bridge Association. If each club holds two games per week and each duplicate game consists of 24 deals, how many deals are played in club games each year?

30. **Ice Cream Selections** An ice cream parlor offers 25 flavors of ice cream. Assuming that order matters, how many different two-scoop cones can be made? What if no flavor can be repeated?

31. **Internet Accounts** A college of 20,000 students provides each student with an Internet account. Explain why letting each student have his or her initials as the username cannot possibly work. Assume that each person has a first name, middle name, and last name and therefore that each person's initials consist of three letters.

32. **Pairs of Initials** A company has 700 employees. Explain why there must be two people with the same first and last initials.

33. **Game Outcomes** The final score in a soccer game is 6 to 4. How many different halftime scores are possible?

34. **Selecting an Outfit** Each day, Gloria dresses in a blouse, a skirt, and shoes. She wants to wear a different combination on every day of the year. If she has the same number of blouses, skirts, and pairs of shoes, how many of each article would she need to have a different combination every day?

35. **Mismatched Gloves** A man has five different pairs of gloves. In how many ways can he select a right-hand glove and a left-hand glove that do not match?

36. **Mismatched Shoes** Fred has 11 different pairs of shoes. In how many ways can he put on a pair of shoes that do not match?

37. **Coin Tosses** Toss a coin six times, and observe the sequence of heads and tails that results. How many different sequences are possible?

38. **Coin Tosses** Refer to Exercise 37. In how many of the sequences are the first and last tosses identical?

39. **Exam Questions** An exam contains five true-or-false questions. In how many different ways can the exam be completed? Assume that every question must be answered.

40. **Exam Questions** An exam contains five true-or-false questions. In how many ways can the exam be completed if leaving the answer blank is also an option?

41. **Exam Questions** Each of the 10 questions on a multiple-choice exam has four possible answers. How many different ways are there for a student to answer the questions? Assume that every question must be answered.

42. **Exam Questions** Rework Exercise 41 under the assumption that not every question must be answered.

ZIP Codes ZIP (Zone Improvement Plan) codes, sequences of five digits, were introduced by the United States Post Office Department in 1963.

43. How many ZIP codes are possible?

44. ZIP codes for Delaware, New York, and Pennsylvania begin with the digit 1. How many such ZIP codes are possible?

45. **Group Pictures** How many ways can eight people stand in a line for a group picture? If you took a picture every 15 seconds (day and night with no breaks), how long would it take to photograph every possible arrangement?

46. **License Plates** A company is manufacturing license plates with the pattern LL#-##LL, where L represents a letter and # represents a digit from 1 through 9. If a letter can be any letter from A to Z except O, how many different license plates are possible? If the company produces 500,000 license plates per week, how many years will be required to make every possible license plate?

47. **Menu Selections** A college student eats all of their meals at a restaurant offering six breakfast specials, seven lunch specials, and four dinner specials. How many days can they go without repeating an entire day's menu selections?

48. **Menu Selections** A restaurant menu lists 7 appetizers, 10 entrées, and 4 desserts. How many ways can a diner select a three-course meal?

49. **Gift Wrapping** The gift-wrap desk at a large department store offers 5 box sizes, 10 wrapping papers, 7 colors of ribbon in 2 widths, and 9 special items to be added on the box. How many different ways are there to package a gift, assuming that the customer must choose at least a box but need not choose any of the other offerings?

50. **Selecting Fruit** José was told to create a gift basket containing one dozen oranges, eight apples, and a half-pound of grapes. When he gets to the store, he finds five varieties of oranges, five varieties of apples, and two varieties of grapes. Assuming

that he selects only one variety of each type of fruit, how many different gift baskets of fruit could he bring home?

51. **Shading of Venn Diagrams** How many different ways can a Venn diagram with two circles be shaded?

52. **Shading of Venn Diagrams** How many different ways can a Venn diagram with three circles be shaded?

Roulette An American roulette wheel consists of 38 numbered pockets. Two of them (numbered 0 and 00) are colored green, 18 are colored red, and 18 are colored black. Gamblers bet on which numbered pocket a ball will fall into when the wheel is spun. For Exercises 53 and 54, assume the wheel is spun three times.

53. How many outcomes are possible if the first number is green?

54. How many outcomes are possible if all three numbers are red and no number repeats?

55. **Batting Orders** The manager of a Little League baseball team has picked the nine starting players for a game. How many different batting orders are possible under each of the following conditions?
    (a) There are no restrictions.
    (b) The pitcher must bat last.
    (c) The pitcher must bat last, the catcher eighth, and the shortstop first.

56. **Test Volunteers** A physiologist wants to test the effects of exercise and meditation on blood pressure. She devises four different exercise programs and three different meditation programs. If she wants 10 subjects for each combination of exercise and meditation program, how many volunteers must she recruit?

57. **Handshakes** Two 10-member basketball teams play a game. After the game, each of the members of the winning team shakes hands once with each member of both teams. How many handshakes take place?

58. **Band Selections** In how many ways can a band play a set of three waltzes and three tangos without repeating any song, such that the first, third, and fifth songs are waltzes?

59. **Colored Houses** Six houses in a row are each to be painted with one of the colors red, blue, green, and yellow. In how many different ways can the houses be painted so that no two adjacent houses are of the same color?

60. **Chair Varieties** A furniture manufacturer makes three types of upholstered chairs and offers 20 fabrics. How many different chairs are available?

61. **Ballots** Seven candidates for mayor, four candidates for city council president, and six propositions are being put before the electorate. How many different ballots could be cast, assuming that every voter votes on each of the items? If voters can choose to leave any item blank, how many different ballots are possible?

62. **College Applications** Allison is preparing her applications for college. She will apply to three community colleges and has to fill out six parts in each of those applications. She will apply to three four-year schools, each of which has a seven-part application. How many application segments must she complete?

63. **Paths to Texas** Consider the triangular display of letters below. Start with the letter T at the top and move down the triangle to a letter S at the bottom. From any given letter, move only to one of the letters directly below it on the left or right. How many different paths spell $TEXAS$?

$$
\begin{array}{ccccccccc}
 & & & & T & & & & \\
 & & & E & & E & & & \\
 & & X & & X & & X & & \\
 & A & & A & & A & & A & \\
S & & S & & S & & S & & S
\end{array}
$$

64. **Railroad Tickets** A railway has 20 stations. If the names of the point of departure and the destination are printed on each ticket, how many different kinds of single tickets must be printed? How many different kinds of tickets are needed if each ticket may be used in either direction between two stations?

Solutions to Check Your Understanding 4

1. 72. Pretend that you are given the task of seating the five people. This task consists of five choices performed consecutively, as shown in Table 1. After you have performed choice 1, four people will remain, and any one of these four can be seated in the right front seat. After choice 2, three people remain, and so on. By the generalized multiplication principle, the task can be performed in $3 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 72$ ways.

2. 3^{10}. The task of answering the questions consists of 10 consecutive choices, each of which can be performed in three ways. Therefore, by the generalized multiplication principle, the task can be performed in

$$\underbrace{3 \cdot 3 \cdot 3 \cdot \cdots \cdot 3}_{10 \text{ terms}} \quad \text{ways.}$$

Note: The answer can be left as 3^{10} or can be multiplied out to 59,049.

Table 1

Choice	Number of Ways Choice Can Be Performed
1: Select person to drive.	3
2: Select person for right front seat.	4
3: Select person for left rear seat.	3
4: Select person for middle rear seat.	2
5: Select person for right rear seat.	1

5 Permutations and Combinations

In preceding sections, we have solved a variety of counting problems by using Venn diagrams and the generalized multiplication principle. Let us now turn our attention to two types of counting problems that occur very frequently and that can be solved by using formulas derived from the generalized multiplication principle. These problems involve what are called *permutations* and *combinations*, which are particular types of arrangements of elements of a set. The sorts of arrangements that we have in mind are illustrated in two problems:

Problem A How many words (by which we mean *strings of letters*) of two distinct letters can be formed from the letters $\{a, b, c\}$?

Problem B A construction crew has three members. A team of two must be chosen for a particular job. In how many ways can the team be chosen?

Each of the two problems can be solved by enumerating all possibilities.

Solution of Problem A There are six possible words, namely,

$$ab \qquad ac \qquad ba \qquad bc \qquad ca \qquad cb.$$

Solution of Problem B Designate the three crew members by a, b, and c. Then there are three possible two-person teams, namely,

$$\{a, b\} \quad \{a, c\} \quad \{b, c\}.$$

Note that $\{b, a\}$, the team consisting of b and a, is the same as the team $\{a, b\}$.

We deliberately set up both problems with the same letters in order to facilitate comparison. Both problems are concerned with counting the numbers of arrangements of the elements of the set $\{a, b, c\}$, taken two at a time, without allowing repetition. (For example, aa was not allowed.) However, in Problem A, the order of the arrangement mattered, whereas in Problem B it did not. Arrangements of the sort considered in Problem A are called *permutations*, whereas those in Problem B are called *combinations*.

More precisely, suppose that we are given a set of n distinguishable objects.

> **Permutations** A **permutation of n objects taken r at a time** is an arrangement of r of the n objects in a specific order.

So, for example, Problem A was concerned with permutations of the three objects a, b, c $(n = 3)$ taken two at a time $(r = 2)$.

> **Combinations** **A combination of n objects taken r at a time** is a selection of r objects from among the n, with order disregarded.

Thus, for example, in Problem B, we considered combinations of the three objects a, b, c $(n = 3)$ taken two at a time $(r = 2)$.

It is convenient to introduce the notation that follows for counting permutations and combinations. Let

$$P(n, r) = \text{the number of permutations of } n \text{ objects taken } r \text{ at a time}$$
$$C(n, r) = \text{the number of combinations of } n \text{ objects taken } r \text{ at a time.}$$

Thus, for example, from our solutions to Problems A and B, we have

$$P(3, 2) = 6 \qquad C(3, 2) = 3.$$

Very simple formulas for $P(n, r)$ and $C(n, r)$ allow us to calculate these quantities for any n and r. Let us begin by stating the formula for $P(n, r)$. For $r = 1, 2, 3$, respectively,

$$P(n, 1) = n$$
$$P(n, 2) = n(n-1) \qquad \text{(two factors)},$$
$$P(n, 3) = n(n-1)(n-2) \qquad \text{(three factors)}.$$

Continuing, we obtain the following formula:

Permutation Formula The number of permutations of n objects taken r at a time, $P(n, r)$, is the product of the r whole numbers counting down by 1 from n. That is,

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \qquad (r \text{ factors}), \tag{1}$$

or

$$P(n, r) = \frac{n!}{(n-r)!} \tag{2}$$

This formula is verified at the end of the section.

EXAMPLE 1 **Applying the Permutation Formula** Compute the following numbers:

(a) $P(100, 2)$ **(b)** $P(6, 4)$ **(c)** $P(5, 5)$

SOLUTION **(a)** Here, $n = 100$, $r = 2$. So we take the product of two factors, beginning with 100:

$$P(100, 2) = 100 \cdot 99 = 9900.$$

(b) $P(6, 4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360$

(c) $P(5, 5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

» Now Try Exercise 1

Combination Formula The number of combinations of n objects taken r at a time, $C(n, r)$, is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \qquad (r \text{ factors}), \tag{3}$$

or

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

This formula is verified at the end of this section.

EXAMPLE 2 **Applying the Combination Formula** Compute the following numbers:

(a) $C(100, 2)$ **(b)** $C(6, 4)$ **(c)** $C(5, 5)$

SOLUTION **(a)** $C(100, 2) = \dfrac{P(100, 2)}{2!} = \dfrac{100 \cdot 99}{2 \cdot 1} = 4950$

(b) $C(6, 4) = \dfrac{P(6, 4)}{4!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3}{4 \cdot 3 \cdot 2 \cdot 1} = 15$

(c) $C(5, 5) = \dfrac{P(5, 5)}{5!} = \dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1$

» Now Try Exercise 5

EXAMPLE 3 **Applying the Permutation and Combination Formulas** Solve Problems A and B, using formulas (1) and (3).

SOLUTION The number of two-letter words that can be formed from the three letters a, b, and c is equal to $P(3, 2) = 3 \cdot 2 = 6$, in agreement with our previous solution.

The number of two-worker teams that can be formed from three individuals is equal to $C(3, 2)$, and

$$C(3, 2) = \frac{P(3, 2)}{2!} = \frac{3 \cdot 2}{2 \cdot 1} = 3,$$

in agreement with our previous result. **»** ***Now Try Exercises 27 and 29***

EXAMPLE 4 **Selecting a Committee** The board of directors of a corporation has 10 members. In how many ways can they choose a committee of three board members to negotiate a merger?

SOLUTION Since the committee of three involves no ordering of its members, we are concerned here with combinations. The number of combinations of 10 people taken 3 at a time is $C(10, 3)$, which is

$$C(10, 3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120.$$

Thus, there are 120 possibilities for the committee. **»** ***Now Try Exercise 31***

EXAMPLE 5 **Selecting Club Officers** A club has 10 members. In how many ways can they choose a slate of four officers, consisting of a president, vice president, secretary, and treasurer?

SOLUTION In this problem, we are dealing with an ordering of four members. (The first is the president, the second the vice president, and so on.) So we are dealing with permutations, and the number of ways of choosing the officers is

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040.$$

» ***Now Try Exercise 33***

EXAMPLE 6 **Outcomes of a Horse Race** Eight horses are entered in a race in which a first, second, and third prize will be awarded. Assuming no ties, how many different outcomes are possible?

SOLUTION In this example, we are considering ordered arrangements of three horses, so we are dealing with permutations. The number of permutations of eight horses taken three at a time is

$$P(8, 3) = 8 \cdot 7 \cdot 6 = 336,$$

so the number of possible outcomes of the race is 336. **»** ***Now Try Exercise 39***

EXAMPLE 7 **Polling Sample** A political pollster wishes to survey 1500 individuals chosen from a sample of 5,000,000 adults. In how many ways can the 1500 individuals be chosen?

SOLUTION No ordering of the 1500 individuals is involved, so we are dealing with combinations. So the number in question is $C(5{,}000{,}000, 1500)$, a number too large to be written down in digit form. (It has several thousand digits!) But it could be calculated with the aid of a computer. **»** ***Now Try Exercise 29***

EXAMPLE 8 **Seating Arrangements** Three couples go on a movie date. In how many ways can they be seated in a row of six seats so that each couple is seated together?

SOLUTION A seating arrangement can be composed of four consecutive choices:

Select an order in which to seat the couples	Seat the left-most couple	Seat the middle couple	Seat the right-most couple
Choice 1	Choice 2	Choice 3	Choice 4

Since there are three couples, choice 1 can be performed in 3! ways. After the order of the three couples has been chosen, there are 2! ways to seat the two members of the leftmost couple. Similarly, there are 2! ways to perform choice 3 and 2! ways to perform choice 4.

By the generalized multiplication principle, the number of possible arrangements is $3! \cdot 2! \cdot 2! \cdot 2! = 48$. **>> Now Try Exercise 59**

EXAMPLE 9 **Arranging Books** If you have six books, in how many ways can you select four books and arrange them on a shelf?

SOLUTION Arrange four of the six books in order. This can be done in $P(6, 4) = 360$ ways.

As is often the case in counting problems, there are multiple ways to arrive at a solution.

ALTERNATIVE SOLUTION First, select the four books from the group of six. Since order does not matter at this point, this can be done in $C(6, 4)$ ways. Next, arrange the selection of four books on the shelf, which can be done in 4! ways. By the multiplication principle, the number of possible arrangements is $C(6, 4) \cdot 4! = 15 \cdot 24 = 360$. **>> Now Try Exercise 58**

Example 9 shows that $P(n, r) = C(n, r) \cdot r!$ when $n = 6$ and $r = 4$. The discussion that follows shows that this formula holds for all values of n and r.

Verification of the Formulas for $P(n, r)$ and $C(n, r)$

Let us first derive the formula for $P(n, r)$, the number of permutations of n objects taken r at a time. The task of choosing r objects (in a given order) consists of r consecutive choices (Fig. 1). The first choice can be performed in n ways. For each way that the first choice is performed, one object will have been used up so that we can perform the second choice in $n - 1$ ways, and so on. For each way of performing the sequence of choices $1, 2, 3, \ldots, r - 1$, the rth choice can be performed in $n - (r - 1) = n - r + 1$ ways. By the generalized multiplication principle, the task of choosing the r objects from among the n can be performed in

$$n(n-1) \cdot \cdots \cdot (n - r + 1) \quad \text{ways.}$$

That is,

$$P(n, r) = n(n-1) \cdot \cdots \cdot (n - r + 1),$$

which is formula (1).

Choose 1st object	Choose 2nd object	. . .	Choose rth object

Figure 1

Let us now verify the formula for $C(n, r)$, the number of combinations of n objects taken r at a time. Each such combination is a set of r objects and, therefore, can be ordered in

$$P(r, r) = r(r-1) \cdot \cdots \cdot 2 \cdot 1 = r!$$

ways by formula (1). In other words, each different combination of r objects gives rise to $r!$ permutations of the same r objects. On the other hand, each permutation of n objects taken r at a time gives rise to a combination of n objects taken r at a time, by simply ignoring the order of the permutation. Thus, if we start with the $P(n, r)$ permutations, we will have all of the combinations of n objects taken r at a time, with each combination repeated $r!$ times. Thus,

$$P(n, r) = r!\, C(n, r).$$

On dividing both sides of the equation by $r!$, we obtain formula (3).

INCORPORATING TECHNOLOGY

Computing Permutations, Combinations, and Factorials Most graphing calculators have commands to compute $P(n, r)$, $C(n, r)$, and $n!$. For instance, the MATH key on the TI-84 Plus leads to the PROB menu of Fig. 2, which contains the commands **nPr**, **nCr**, and **!**. Figure 3 shows how these commands are used. *Note:* The number **8.799226775E15** represents $8.799226775 \times 10^{15}$.

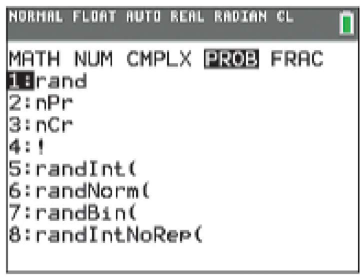


Figure 2

Figure 3

The values of $n!$, $P(n, r)$, and $C(n, r)$ are calculated in an Excel spreadsheet with the functions FACT(n), PERMUT(n, r), and COMBIN(n, r).

WolframAlpha Factorials, permutations, and combinations can be calculated in Wolfram|Alpha with the instructions ***n*!**, **P(*n*, *r*)**, and **C(*n*, *r*)**. Some alternative instructions are **factorial *n***, **permutations (*n*, *r*)**, and **combinations (*n*, *r*)** or ***n* choose *r***.

Check Your Understanding 5

Solutions can be found following the section exercises.

1. Calculate the following values:
   (a) $5!$ **(b)** $P(5, 5)$ **(c)** $P(7, 3)$ **(d)** $C(7, 3)$

2. A newborn child is to be given a first name and a middle name from a selection of 10 names. How many different possibilities are there?

EXERCISES 5

For Exercises 1–20, calculate the values.

1. $P(4, 2)$	**2.** $P(5, 1)$	**3.** $P(6, 3)$
4. $P(5, 4)$	**5.** $C(10, 3)$	**6.** $C(12, 2)$
7. $C(5, 4)$	**8.** $C(6, 3)$	**9.** $P(7, 1)$
10. $P(5, 5)$	**11.** $P(n, 1)$	**12.** $P(n, 2)$
13. $C(4, 4)$	**14.** $C(n, 2)$	**15.** $C(n, n-2)$
16. $C(n, 1)$	**17.** $6!$	**18.** $\frac{10!}{4!}$
19. $\frac{9!}{7!}$	**20.** $7!$	

In Exercises 21–26, determine whether the computation involves a permutation, a combination, or neither.

21. Stock Abbreviations The number of different stock abbreviations for which each abbreviation consists of four letters, none repeated.

22. Airport Codes The number of different airport codes in which each code consists of three letters, none repeated.

23. Ice Cream Flavors The selection of three different flavors of ice cream (out of 29 flavors) for the three scoops of ice cream in a sundae.

24. Ice Cream Flavors The selection of three different flavors of ice cream (out of 29 flavors) for the three scoops of ice cream on an ice cream cone, where order matters.

25. Rolling Dice The number of possible sums when two dice are rolled and the numbers displayed are added.

26. Meal Choices The number of possible meals consisting of an appetizer, a main course, and a dessert from a restaurant that offers 5 different appetizers, 10 main courses, and 4 desserts.

27. **Group Picture** In how many ways can four people line up in a row for a group picture?

28. **Waiting in Line** In how many ways can six people line up at a single counter to order food at a fast-food restaurant?

29. **Book Selection** How many different selections of seven books can be made from nine books?

30. **Pizza Varieties** A pizzeria offers five toppings for the plain cheese base of the pizzas. How many pizzas can be made that use three different toppings?

31. **Selecting Colleges** A high school student decides to apply to four of the eight Ivy League colleges. In how many possible ways can the four colleges be selected?

32. **Senate Committees** In how many different ways can a committee of 5 senators be selected from the 100 members of the U.S. Senate?

33. **Ranking Teams** A sportswriter makes a preseason guess of the top five football teams (in order) from among the 65 Power Five conference teams. How many different possibilities are there?

34. **CD Changer** Suppose that you have 36 CDs and your CD player has five slots numbered 1 through 5. How many ways can you fill your CD player?

35. **Guest Lists** How many ways can you choose five out of 10 friends to invite to a dinner party?

36. **Choosing Exam Questions** A student is required to work exactly four problems from an eight-problem exam. In how many ways can the problems be chosen?

37. **DVDs** In a batch of 100 DVDs, seven are defective. A sample of three DVDs is to be selected from the batch. How many samples are possible? How many of the samples consist of all defective DVDs?

38. **Debates** There are 17 candidates for an elected office. If 10 candidates are selected to participate in a debate, determine the total number of possible debate groups.

39. **Race Winners** Theoretically, assuming no ties, how many possibilities are there for first, second, and third places in a marathon race with 150 entries?

40. **Player Introductions** The five starting players of a basketball team are introduced one at a time. In how many different ways can they be introduced?

Poker Hands Exercises 41–44 refer to poker hands. A poker hand consists of 5 cards selected from a standard deck of 52 cards.

41. How many different poker hands are there?

42. How many different poker hands consist entirely of aces and kings?

43. How many different poker hands consist entirely of clubs?

44. How many different poker hands consist entirely of red cards?

45. **Distributing Sandwiches** Five students order different sandwiches at a campus eatery. The waiter forgets who ordered what and gives out the sandwiches at random. In how many different ways can the sandwiches be distributed?

46. **Nautical Signals** A nautical signal consists of three flags arranged vertically on a flagpole. If a sailor has six flags, each of a different color, how many different signals are possible?

47. **Selecting Sweaters** Suppose that you own 10 sweaters.
    (a) How many ways can you select four of them to take on a trip?
    (b) How many ways can you select six of the sweaters to leave at home?
    (c) Explain why the answers to parts (a) and (b) are the same.

48. **Distributing Books** Fred has 12 different books.
    (a) Suppose that Fred first gives three books to Jill and then gives four of the remaining books to Jack. How many different outcomes are possible?
    (b) Suppose that Fred first gives four books to Jack and then gives three of the remaining books to Jill. How many different outcomes are possible?
    (c) Explain why the answers to parts (a) and (b) are the same.

49. **Conference Games** In an eight-team football conference, each team plays every other team exactly once. How many games must be played?

50. **League Games** In a six-team softball league, each team plays every other team three times during the season. How many games must be scheduled?

51. **Powerball** In the Powerball lottery, five white balls are drawn out of a drum with 69 numbered white balls and then one red ball is drawn out of a drum with 26 numbered red balls. The jackpot is won by guessing all five white balls in any order and the red Powerball. Determine the number of possible outcomes.

52. **Baseball Lineup** On a children's baseball team, there are four players who can play at any of the following infield positions: catcher, first base, third base, and shortstop. There are five possible pitchers, none of whom plays any other position. And there are four players who can play any of the three outfield positions (right, left, and center) or second base. In how many ways can the coach assign players to positions?

Lotto Exercises 53 and 54 refer to the New York State lottery (Lotto). When it was first established, a contestant had to select six numbers from 1 to 49. A few years later, the numbers 50 through 54 were added. In March 2007, the numbers 55 through 59 were added.

53. The number of possible combinations of six numbers selected from 1 to 59 is approximately _____ times the number of combinations selected from 1 through 49.
    (a) 2 (b) 3 (c) 10 (d) 100

54. Drawings for Lotto are held twice per week. Suppose that you decide to purchase 110 tickets for each drawing and never use the same combination twice. Approximately how many years would be required before you would have bet on every possible combination?
    (a) 100 (b) 1000 (c) 2000 (d) 4000

55. **Choosing Candy** Two children, Moe and Joe, are allowed to select candy from a plate of nine pieces of candy. Moe, being younger, is allowed to choose first but can take only two candies. Joe is then allowed to take four of the remaining candies. Joe complains that he has fewer options than Moe. Is Joe correct? How many options will each child have?

56. **Committee Selection** The 12 members of the Gotham City Council consists of four members from each of the city's three wards. In how many ways can a committee of six council

members be selected if the committee must contain at least one council member from each ward?

57. **Group Picture** The student council at Gotham College is made up of four freshmen, five sophomores, six juniors, and seven seniors. A yearbook photographer would like to line up three council members from each class for a picture. How many different pictures are possible if each group of classmates stands together?

58. **Arranging Books** George has three books by each of his five favorite authors. In how many ways can the books be placed on a shelf if books by the same author must be together?

59. **Seating Arrangements** In the quiz show *It's Academic*, three-person teams from three high schools are seated in a row, with each team seated together. How many different seating arrangements are possible?

60. **Displaying Paintings** An art gallery has four paintings, by each of three artists, hanging in a row, with paintings by the same artist grouped together. How many different arrangements are possible?

61. **Handshakes** At a party, everyone shakes hands with everyone else. If 45 handshakes take place, how many people are at the party?

62. **Football Games** In a football league, each team plays one game against each other team in the league. If 55 games are played, how many teams are in the league?

63. **Side Dishes** A restaurant offers its customers a choice of three side dishes with each meal. The side dishes can be chosen from a list of 15 possibilities, with duplications allowed. For instance, a customer can order two sides of mashed potatoes and one side of string beans. Show that there are 680 possible options for the three side dishes.

64. **Ice Cream Specials** An ice cream parlor offers a special consisting of three scoops of ice cream chosen from 16 different flavors. Duplication of flavors is allowed. For instance, one possibility is two scoops of chocolate and one scoop of vanilla. Show that there are 816 different possible options for the special.

65. **Determine Position** There are $6! = 720$ six-letter words (that is, strings of letters) that can be made from the letters C, N, O, S, T, and U. If these 720 words are listed in alphabetical order, what position in the list will be occupied by TUCSON? *Hint:* Count the number of words that follow TUCSON in the list. They must each be of the form TUNxxx or Uxxxxx.

66. **Detour-Prone ZIP Codes** A five-digit ZIP code is said to be detour prone if it looks like a valid and different ZIP code when read upside down (Fig. 4). For instance, 68901 and 88111 are detour prone, whereas 32145 and 10801 are not. How many of the 10^5 possible ZIP code numbers are detour prone?

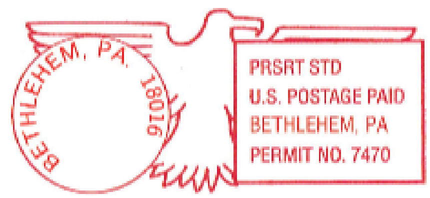


Figure 4

TECHNOLOGY EXERCISES

67. **Lottery**
   (a) Calculate the number of possible lottery tickets if the player must choose five distinct numbers from 0 to 44, inclusive, where the order does not matter. The winner must match all five.
   (b) Calculate the number of lottery tickets if the player must choose four distinct numbers from 0 to 99, inclusive, where the order does not matter. The winner must match all four.
   (c) In which lottery does the player have a better chance of choosing the randomly selected winning numbers?

68. **Bridge** A bridge hand contains 13 cards.
   (a) What percent of bridge hands contain all four aces?
   (b) What percent of bridge hands contain the two red kings, the two red queens, and no other kings or queens?
   (c) Which is more likely—a bridge hand with four aces or one with the two red kings, the two red queens, and no other kings or queens?

69. **Cards versus Atoms** Are there more ways to order a deck of 52 cards than there are atoms on Earth? *Note:* There are about 10^{50} atoms on Earth.

70. **Alphabet versus Atoms** Are there more ways to rearrange the 26 letters of the alphabet than there are atoms on Earth?

Solutions to Check Your Understanding 5

1. (a) $5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$
   (b) $P(5,5) = 5\cdot4\cdot3\cdot2\cdot1 = 120$
   [$P(n,n)$ is the same as $n!$]
   (c) $P(7,3) = \underbrace{7\cdot6\cdot5}_{3 \text{ factors}} = 210$
   [$P(7,3)$ is the product of the 3 whole numbers, counting down by 1 from 7.]
   (d) $C(7,3) = \dfrac{7\cdot6\cdot5}{3\cdot2\cdot1} = \dfrac{7\cdot\cancel{6}\cdot5}{\cancel{3}\cdot\cancel{2}\cdot1} = 35$
   [A convenient procedure to follow when calculating $C(n,r)$ is first to write the product expansion of $r!$ in the denominator and then to write in the numerator a whole number above each integer in the denominator. The whole numbers should begin with n and successively decrease by 1.]

2. 90. The first question to be asked here is whether permutations or combinations are involved. Two names are to be selected, and the order of the names is important. (The name Amanda Beth is different from the name Beth Amanda.) Since the problem asks for arrangements of 10 names taken 2 at a time in a *specific order*, the number of arrangements is $P(10,2) = 10\cdot9 = 90$. In general, order is important if a different outcome results when two items in the selection are interchanged.

6 Further Counting Techniques

In Section 5, we introduced permutations and combinations and developed formulas for counting all permutations (or combinations) of a given type. Many counting problems can be formulated in terms of permutations or combinations. But to use the formulas of Section 5 successfully, we must be able to recognize these problems when they occur and to translate them into a form in which the formulas may be applied. In this section, we practice doing that. We consider five typical applications giving rise to permutations or combinations. At first glance, the first two applications may seem to have little practical significance. However, they suggest a common way to "model" outcomes of real-life situations having two equally likely results.

As our first application, consider a coin-tossing experiment in which we toss a coin a fixed number of times. We can describe the outcome of the experiment as a sequence of "heads" and "tails." For instance, if a coin is tossed three times, then one possible outcome is "heads on the first toss, tails on the second toss, and tails on the third toss." This outcome can be abbreviated as HTT. We can use the methods of the preceding sections to count the number of possible outcomes having various prescribed properties.

EXAMPLE 1 **Tossing a Coin Ten Times** Suppose that an experiment consists of tossing a coin 10 times and observing the sequence of heads and tails.

(a) How many different outcomes are possible?

(b) How many different outcomes have exactly four heads?

SOLUTION **(a)** Visualize each outcome of the experiment as a sequence of 10 boxes, where each box contains one letter, H or T, with the first box recording the result of the first toss, the second box recording the result of the second toss, and so forth.

$$\boxed{H}\ \boxed{T}\ \boxed{H}\ \boxed{T}\ \boxed{T}\ \boxed{T}\ \boxed{H}\ \boxed{T}\ \boxed{H}\ \boxed{T}$$
$$1\quad 2\quad 3\quad 4\quad 5\quad 6\quad 7\quad 8\quad 9\quad 10$$

Each box can be filled in two ways. So by the generalized multiplication principle, the sequence of 10 boxes can be filled in

$$\underbrace{2\cdot 2\cdot \cdots \cdot 2}_{10\text{ factors}} = 2^{10}$$

ways. So there are $2^{10} = 1024$ different possible outcomes.

(b) An outcome with 4 heads corresponds to filling the boxes with 4 H's and 6 T's. A particular outcome is determined as soon as we decide where to place the H's. The 4 boxes to receive H's can be selected from the 10 boxes in $C(10, 4)$ ways. So the number of outcomes with 4 heads is

$$C(10,4) = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210.$$

>> Now Try Exercise 1

Ideas similar to those applied in Example 1 are useful in counting even more complicated sets of outcomes of coin-tossing experiments. The second part of our next example highlights a technique that can often save time and effort.

EXAMPLE 2 **Tossing a Coin Ten Times** Consider the coin-tossing experiment of Example 1.

(a) How many different outcomes have at most two heads?

(b) How many different outcomes have at least three heads?

SOLUTION **(a)** The outcomes with at most two heads are those having 0, 1, or 2 heads. Let us count the number of these outcomes separately:

0 heads: There is 1 outcome, namely, T T T T T T T T T T.

1 head: To determine such an outcome, we just select the box in which to put the single H. And this can be done in $C(10, 1) = 10$ ways.

2 heads: To determine such an outcome, we just select the boxes in which to put the two H's. And this can be done in $C(10, 2) = (10 \cdot 9)/(2 \cdot 1) = 45$ ways.

Adding up all the possible outcomes, we see that the number of outcomes with at most two heads is $1 + 10 + 45 = 56$.

(b) "At least three heads" refers to an outcome with either 3, 4, 5, 6, 7, 8, 9, or 10 heads. The total number of such outcomes is

$$C(10, 3) + C(10, 4) + \cdots + C(10, 10).$$

This sum can, of course, be calculated, but there is a less tedious way to solve the problem. Just start with all outcomes [1024 of them by Example 1(a)], and subtract those with at most two heads [56 of them by part (a)]. So the number of outcomes with at least three heads is $1024 - 56 = 968$. ***>> Now Try Exercise 3***

The solution to part (b) of Example 2 employs a useful counting technique.

Complement Rule for Counting If U is the set of all possible outcomes and S is a subset of U, then S' is the set of all outcomes for which S does *not* occur, and

$$n(S') = n(U) - n(S).$$

Let us now turn to a different sort of counting problem, namely, one that involves counting the number of paths between two points.

EXAMPLE 3 **Routes through a City** In Fig. 1, we have drawn a partial map of the streets in New York City. A tourist wishes to walk from Times Square to Grand Central Station. We have drawn two possible routes. What is the total number of routes (with no back-tracking) from Times Square to Grand Central Station?

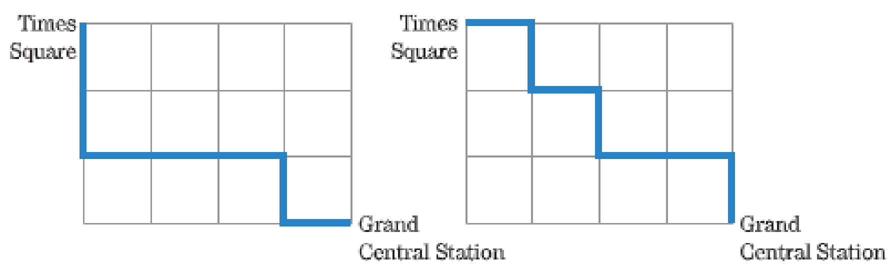


Figure 1

SOLUTION Any particular route can be described by giving the directions of each block walked in the appropriate order. For instance, the route on the left of Fig. 1 is described as "a block south, a block south, a block east, a block east, a block east, a block south, a block east." Using S for south and E for east, this route can be designated by the string of letters SSEEESE. Similarly, the route on the right is ESESEES. Note that each route is then described by a string of seven letters, of which three are S's (we must go three blocks south) and four are E's (we must go four blocks east). Selecting a route is thus the same as placing three S's in a string of seven boxes:

	S		S			S

The three boxes to receive S's can be selected in $C(7, 3) = 35$ ways. So the number of paths from Times Square to Grand Central Station is 35. ***>> Now Try Exercise 11***

EXAMPLE 4 **Routes through a City** Refer to the street map of Example 3. In how many of the routes does the tourist never walk south for two consecutive blocks?

SOLUTION Each route is a string of letters containing three S's and four E's. We are asked to count the number of such strings in which two S's are never adjacent to each other. One way to construct such a string is to write down four E's and then decide where to insert the three S's. The arrows below show the five places that the S's can be inserted:

$$\begin{array}{ccccccccc} & \mathbf{E} & & \mathbf{E} & & \mathbf{E} & & \mathbf{E} & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \end{array}$$

The three insertion points can be selected in $C(5, 3) = 10$ ways. Therefore, there are 10 routes without consecutive souths. **>> *Now Try Exercise 15***

Let us now move on to a third type of counting problem. Suppose that we have an urn in which there are a certain number of red balls and a certain number of white balls. We perform an experiment that consists of selecting a number of balls from the urn and observing the color distribution of the sample selected. (This model may be used, for example, to describe the process of selecting people to be polled in a survey. The different colors would correspond to different opinions.) By using familiar counting techniques, we can calculate the number of possible samples having a given color distribution. The next example illustrates a typical computation.

EXAMPLE 5 **Selecting Balls from an Urn** An urn contains 25 numbered balls, of which 15 are red and 10 are white. A sample of 5 balls is to be selected.

(a) How many different samples are possible?
(b) How many samples contain all red balls?
(c) How many samples contain 3 red balls and 2 white balls?
(d) How many samples contain at least 4 red balls?

SOLUTION **(a)** A sample is just an unordered selection of 5 balls out of 25. There are $C(25, 5)$ such samples. Numerically, we have

$$C(25, 5) = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 53{,}130$$

samples.

(b) To form a sample of all red balls, we must select 5 balls from the 15 red ones. This can be done in $C(15, 5)$ ways—that is, in

$$C(15, 5) = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 3003$$

ways.

(c) To answer this question, we use both the multiplication principle and the formula for $C(n, r)$. We form a sample of 3 red balls and 2 white balls, using a sequence of two choices:

Select 3 red balls	Select 2 white balls
Choice 1	Choice 2

The first choice can be performed in $C(15, 3)$ ways and the second in $C(10, 2)$ ways. Thus, the total number of samples having 3 red and 2 white balls is $C(15, 3) \cdot C(10, 2)$. That is,

$$C(15, 3) = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 455$$

$$C(10, 2) = \frac{10 \cdot 9}{2 \cdot 1} = 45$$

$$C(15, 3) \cdot C(10, 2) = 455 \cdot 45 = 20{,}475.$$

So the number of possible samples is 20,475.

(d) A sample with at least 4 red balls has either 4 or 5 red balls. By part (b), the number of samples with 5 red balls is 3003. Using the same reasoning as in part (c), the number of samples with 4 red balls is $C(15, 4) \cdot C(10, 1) = 1365 \cdot 10 = 13{,}650$. Thus, the total number of samples having at least 4 red balls is $13{,}650 + 3003 = 16{,}653$.

>> Now Try Exercise 25

EXAMPLE 6

Photo Session The nine justices of the United States Supreme Court can be seated in a row for a photo session in 9! different ways. In how many of the choices will Justice Elena Kagan be seated to the left of, but not necessarily beside, Justice John Roberts?

FIRST SOLUTION A possible seating for the justices can be obtained as follows:

1. Select the set of two chairs for Justices Kagan and Roberts. — $C(9, 2)$ possibilities
2. Seat Justice Kagan in the left chair and Justice Roberts in the right chair. — 1 possibility
3. Seat the remaining justices in the unoccupied seven chairs. — 7! possibilities

Therefore, by the multiplication principle, the number of such seatings are

$$C(9, 2) \cdot 1 \cdot 7! = 36 \cdot 1 \cdot 5040 = 181{,}440.$$

SECOND SOLUTION There are a total of $9! = 362{,}880$ possible seatings of the nine justices. In half of them, Justice Kagan will be seated to the left of Justice Roberts. Therefore, the answer to the question is $\frac{1}{2} \cdot 362{,}880 = 181{,}440$.

>> Now Try Exercise 37

EXAMPLE 7

Numbers How many six-digit numbers are there in which the digits strictly decrease when read from left to right? (Two examples are 865,421 and 976,532.)

SOLUTION One way to obtain a six-digit number is to start with the ten-digit number 9876543210 and remove four digits. Since the four digits can be selected in $C(10, 4) = 210$ different ways, there are 210 six-digit numbers whose digits are strictly decreasing.

>> Now Try Exercise 39

NOTE In counting problems, we often compute permutations or combinations as an intermediate step, then add, subtract, or multiply to solve the problem. A useful guideline to determine when to perform which arithmetic operation is:

- **Addition** If a set of outcomes can be expressed as the union of disjoint subsets of outcomes, then we may calculate the number of outcomes in each subset and add [see Example 2(a)].
- **Subtraction** If a set of outcomes can be expressed as the complement of another set, then we may use the complement rule [see Example 2(b)].
- **Multiplication** If a set of outcomes can be expressed as a sequence of choices, then we may use the multiplication principle (see Example 5). «

Check Your Understanding 6

Solutions can be found following the section exercises.

1. **School Board** A newspaper reporter wants an indication of how the 15 members of the school board feel about a certain proposal. She decides to question a sample of 6 of the board members.
   (a) How many different samples are possible?
   (b) Suppose that 10 of the board members support the proposal, and 5 oppose it. How many of the samples reflect the distribution of the board? That is, in how many of the samples do 4 people support the proposal and 2 oppose it?

2. **Free Throws** A basketball player shoots eight free throws and lists the sequence of results of each trial in order. Let S represent *success* and F represent *failure*. Then, for instance, FFSSSSSS represents the outcome of missing the first two shots and hitting the rest.
   (a) How many different outcomes are possible?
   (b) How many of the outcomes have six successes?

EXERCISES 6

1. **Tossing a Coin** An experiment consists of tossing a coin eight times and observing the sequence of heads and tails.
   (a) How many different outcomes are possible?
   (b) How many different outcomes have exactly four heads?

2. **Tossing a Coin** An experiment consists of tossing a coin nine times and observing the sequence of heads and tails.
   (a) How many different outcomes are possible?
   (b) How many different outcomes have exactly two tails?

3. **Tossing a Coin** An experiment consists of tossing a coin seven times and observing the sequence of heads and tails.
   (a) How many different outcomes have at least five heads?
   (b) How many different outcomes have at most four heads?

4. **Tossing a Coin** An experiment consists of tossing a coin six times and observing the sequence of heads and tails.
   (a) How many different outcomes have at most three heads?
   (b) How many different outcomes have four or more heads?

5. **World Series** In the World Series, the American League team ("A") and the National League team ("N") play until one team wins four games. Each sequence of winners can be designated by a sequence of As and Ns. For instance, NAAAA means the National League won the first game and lost the next four games. In how many ways can a series end in seven games? Six games?

6. **World Series** Refer to Exercise 5. How many different sequences are possible?

7. **Game Outcomes** A football team plays 11 games. In how many ways can these games result in five wins, five losses, and one tie?

8. **Game Outcomes** A chess master plays 15 games. In how many ways can these games result in 10 wins, 2 losses, and 3 draws?

Bytes Exercises 9 and 10 refer to computer bytes. A computer *byte* is a string of eight digits, where each digit is either a zero or a one. Two examples are 01001001 and 11001101.

9. How many bytes have exactly five ones?

10. In how many of the bytes with exactly five ones are no two zeroes next to each other?

11. **Routes through City Streets** Refer to the map in Fig. 2. How many shortest routes are there from A to B?

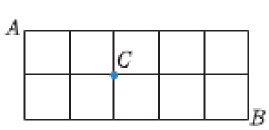


Figure 2

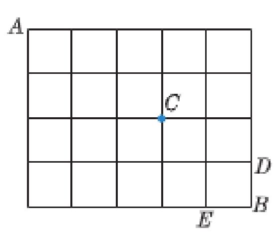


Figure 3

12. **Routes through City Streets** Refer to the map in Fig. 3. How many shortest routes are there from A to B?

13. **Routes through City Streets** Refer to the map in Fig. 3. How many shortest routes are there from A to B that pass through the point C?

14. **Routes through City Streets** Refer to the map in Fig. 2. How many shortest routes are there from A to B that pass through the point C?

15. **Routes through City Streets** Refer to the map in Fig. 2. How many shortest routes are there from A to B that do not have two consecutive souths?

16. **Routes through City Streets** Refer to the map in Fig. 3. How many shortest routes are there from A to B that do not have two consecutive souths?

17. **Routes through City Streets** Refer to the map in Fig. 3. The number of shortest routes from A to B is $C(9, 4)$.
   (a) Observe that the number of shortest routes from A to D is $C(8, 3)$.
   (b) Observe that the number of shortest routes from A to E is $C(8, 4)$.
   (c) By looking at Fig. 3, explain why $C(8, 3) + C(8, 4)$ should equal $C(9, 4)$.
   (d) Calculate the values in part (c) to verify the equality.

18. **Routes through City Streets** Imagine a street map similar to the map in Fig. 3 but having r streets vertically and $n - r$ street horizontally. Then, the number of shortest routes from A to B would be $C(n, r)$.
   (a) Explain why the number of shortest routes A to D, the intersection directly north of B, would be $C(n - 1, r - 1)$.
   (b) Explain why the number of shortest routes A to E, the intersection directly west of B, would be $C(n - 1, r)$.
   (c) Explain why $C(n - 1, r - 1) + C(n + 1, r) = C(n, r)$.

19. **Tossing a Coin** A coin is tossed 10 times, and the sequence of heads and tails is observed. How many of the possible outcomes contain three heads, with no two heads adjacent to each other?

20. **Arranging Books** Four mathematics books and seven history books are arranged on a bookshelf. In how many of the possible arrangements are no two mathematics books next to each other?

21. **Seating Arrangements** In how many ways can six people be seated in a row of ten chairs so that at least two adjacent chairs are vacant? (*Hint*: First use the reasoning in Example 4 to count the number of ways they can be seated so that no two adjacent chairs are empty.)

22. **Photo Session** In 2015, there were three women and six men on the United States Supreme Court. In how many ways could the justices be seated in a row for a group picture where no women sat next to each other?

23. **Selecting Balls from an Urn** An urn contains 12 numbered balls, of which 7 are red and 5 are white. A sample of 5 balls is to be selected.
   (a) How many different samples are possible?
   (b) How many samples contain all red balls?
   (c) How many samples contain two red balls and three white balls?
   (d) How many samples contain at least four red balls?

24. **Selecting Balls from an Urn** An urn contains 15 numbered balls, of which 6 are red and 9 are white. A sample of six balls is to be selected.
   (a) How many different samples are possible?
   (b) How many samples contain all white balls?
   (c) How many samples contain two red balls and four white balls?
   (d) How many samples contain at least two red balls?

25. **Selecting Apples** A bag of 10 apples contains 2 rotten apples and 8 good apples. A shopper selects a sample of three apples from the bag.
   (a) How many different samples are possible?
   (b) How many samples contain all good apples?
   (c) How many samples contain at least one rotten apple?

26. **Selecting Light Bulbs** A package contains 100 LED light bulbs, of which 10 are defective. A sample of five bulbs is selected at random.
   (a) How many different samples are there?
   (b) How many of the samples contain two defective bulbs?
   (c) How many of the samples contain at least one defective bulb?

27. **Subcommittee Selection** A committee has four male and six female members. In how many ways can a subcommittee consisting of two males and two females be selected?

28. **Investment Portfolio** In how many ways can an investor put together a portfolio of five stocks and six bonds selected from her favorite nine stocks and seven bonds?

Poker Hands Exercises 29–32 refer to poker hands. A poker hand consists of 5 cards selected from a standard deck of 52 cards.

29. How many poker hands consist of three aces and two kings?

30. How many poker hands consist of two aces, two cards of another rank, and one card of a third rank?

31. How many poker hands consist of three cards of one rank and two cards of another rank? (Such a poker hand is called a "full house.")

32. How many poker hands consist of two cards of one rank, two cards of another (different) rank, and one card of a third rank? (Such a poker hand is called "two pairs.")

In Exercises 33–36, a "word" is interpreted to be a sequence of letters.

33. **SEQUOIA** How many seven-letter words with no repeated letters contain all five vowels?

34. **FACETIOUS** How many nine-letter words with no repeated letters contain the five vowels in alphabetical order?

35. **ABSTEMIOUS** How many 10-letter words with no repeated letters contain the five vowels in alphabetical order?

36. **DIALOGUE** How many eight-letter words with no repeated letters contain all five vowels?

37. **Photo Session** In 2015, there were three women and six men on the United States Supreme Court. In how many ways could the justices be seated in a row for a group picture in which the three women sat next to each other?

38. **Seating Arrangements** You, a friend, and four other people are to be seated in a row of six chairs. How many arrangements are there in which you and your friend are seated next to each other?

39. **Numbers** In how many five-digit numbers (without zeros) are the digits strictly increasing when read from left to right?

40. **Alphabetical Order** In how many four-letter words (including nonsense words) using four different letters from *A* through *J* are the letters in alphabetical order?

41. **License Plates** Suppose that license plates from a certain state consist of four different letters followed by three different digits. In how many license plates are the letters in alphabetical order and the digits in increasing order?

42. **License Plates** Suppose that license plates from a certain state consist of two different letters followed by four different digits. In how many license plates are the letters in alphabetical order and the digits in increasing order?

43. **Seating Arrangements** A family has 12 members. In how many ways can six family members be seated in a row so that their ages increase from left to right?

44. **Arranging Books** In how many ways can five books out of eight be selected and lined up on a bookshelf so that their page counts increase from left to right? *Note:* Assume that no two books have the same page count.

TECHNOLOGY EXERCISES

45. **Tossing a Coin** What percent of the possible outcomes resulting from tossing a coin 100 times contain exactly 50 heads?

46. **Tossing a Coin** What percent of the possible outcomes resulting from tossing a coin 200 times contain exactly 100 heads?

47. **Selecting Balls from an Urn** Suppose that a sample of 20 balls is selected from an urn containing 50 white balls and 50 red balls. What percent of the possible outcomes contains 10 white balls and 10 red balls?

48. **Selecting Balls from an Urn** Suppose that a sample of 20 balls is selected from an urn containing 100 white balls and 100 red balls. What percent of the possible outcomes contains 10 white balls and 10 red balls?

Solutions to Check Your Understanding 6

1. (a) $C(15, 6)$. Each sample is an unordered selection of 15 objects taken 6 at a time.
   (b) $C(10, 4) \cdot C(5, 2)$. Asking for the number of samples of a certain type is the same as asking for the number of ways that the task of forming such a sample can be performed. This task is composed of two consecutive choices. Choice 1, selecting 4 people from among the 10 who support the proposal, can be performed in $C(10, 4)$ ways. Choice 2, selecting two people from among the five people who oppose the proposal, can be performed in $C(5, 2)$ ways. Therefore, by the multiplication principle, the complete task can be performed in $C(10, 4) \cdot C(5, 2)$ ways.

   Note: $C(15, 6) = 5005$ and $C(10, 4) \cdot C(5, 2) = 2100$. Therefore, less than half of the possible samples reflect the true distribution of the school board.

2. (a) 2^8, or 256. Apply the generalized multiplication principle.
   (b) $C(8, 6)$, or 28. Each outcome having six successes corresponds to a sequence of eight letters of which six are S's and two are F's. Such an outcome is specified by selecting the six locations for the S's from among the eight locations, and this has $C(8, 6)$ possibilities.

7 The Binomial Theorem

In Sections 5 and 6, we dealt with permutations and combinations and, in particular, derived a formula for $C(n, r)$, the number of combinations of n objects taken r at a time. Namely, we have

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n(n-1)\cdot \cdots \cdot (n-r+1)}{r!}. \tag{1}$$

Actually, formula (1) was verified in case both n and r are positive integers. But it is useful to consider $C(n, r)$ also in case $r = 0$. In this case, we are considering the number of combinations of n things taken 0 at a time. There is clearly only one such combination: the one containing no elements. Therefore,

$$C(n, 0) = 1.$$

In Section 5, we encountered another formula for $C(n, r)$:

$$C(n, r) = \frac{n!}{r!(n-r)!}. \tag{2}$$

Note that, for $r = 0$, formula (2) reads

$$C(n, 0) = \frac{n!}{0!\,(n-0)!} = \frac{n!}{0!\,n!} = \frac{1}{0!}.$$

As stated in Section 4, the value of 0! is 1. Then the right-hand side of the preceding equation is 1, so formula (2) also holds for $r = 0$.

Formula (2) can be used to prove many facts about $C(n, r)$. For example, the following formula is useful in calculating $C(n, r)$ for large values of r:

$$C(n, r) = C(n, n-r). \tag{3}$$

EXAMPLE 1 **Applying Formula (3)** Calculate $C(100, 98)$.

SOLUTION If we apply formula (3), we have

$$C(100, 98) = C(100, 100-98) = C(100, 2) = \frac{100 \cdot 99}{2 \cdot 1} = 4950.$$

>> Now Try Exercise 1

Verification of Formula (3) Apply formula (2) to evaluate $C(n, n-r)$:

$$C(n, n-r) = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!}$$
$$= C(n, r) \quad \text{[by formula (2) again]}.$$

The formula is intuitively reasonable, since each time we select a subset of r elements, we are selecting a subset of $n - r$ elements to be excluded. Thus, there are as many subsets of $n - r$ elements as there are subsets of r elements. <<

An alternative notation for $C(n, r)$ is $\binom{n}{r}$. Thus, for example,

$$\binom{5}{2} = C(5, 2) = \frac{5 \cdot 4}{2 \cdot 1} = 10.$$

The symbol $\binom{n}{r}$ is called a **binomial coefficient**. To discover why, let us tabulate the values of $\binom{n}{r}$ for some small values of n and r.

$$n = 2: \quad \binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$$

$$n = 3: \quad \binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$$

$$n = 4: \quad \binom{4}{0} = 1 \quad \binom{4}{1} = 4 \quad \binom{4}{2} = 6 \quad \binom{4}{3} = 4 \quad \binom{4}{4} = 1$$

$$n = 5: \quad \binom{5}{0} = 1 \quad \binom{5}{1} = 5 \quad \binom{5}{2} = 10 \quad \binom{5}{3} = 10 \quad \binom{5}{4} = 5 \quad \binom{5}{5} = 1$$

Each row consists of the coefficients that arise in expanding $(x + y)^n$. To see this, inspect the results of expanding $(x + y)^n$ for $n = 2, 3, 4$, and 5:

$$\begin{aligned}
(x + y)^2 &= x^2 + 2xy + y^2 \\
(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x + y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.
\end{aligned}$$

Compare the coefficients in any row with the values in the corresponding row of binomial coefficients. Note that they are the same. Thus, we see that the binomial coefficients arise as coefficients in multiplying out powers of the binomial $x + y$—hence the name *binomial coefficient*.

What we observed for the exponents $n = 2, 3, 4$, and 5 holds true for any positive integer n. We have the following result, a proof of which is given at the end of this section:

Binomial Theorem

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

EXAMPLE 2

Applying the Binomial Theorem Expand $(x + y)^6$.

SOLUTION By the binomial theorem,

$$\begin{aligned}
(x + y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 \\
+ \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + \binom{6}{6}y^6.
\end{aligned}$$

Furthermore,

$$\binom{6}{0} = 1 \qquad \binom{6}{1} = \frac{6}{1} = 6 \qquad \binom{6}{2} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20 \qquad \binom{6}{4} = \binom{6}{2} = 15$$

$$\binom{6}{5} = \binom{6}{1} = 6 \qquad \binom{6}{6} = \binom{6}{0} = 1.$$

Thus,

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

» Now Try Exercise 21

The binomial theorem can be used to count the number of subsets of a set, as shown in the next example.

EXAMPLE 3 **Counting the Number of Subsets of a Set** Determine the number of subsets of a set with five elements.

SOLUTION Let us count the number of subsets of each possible size. A subset of r elements can be chosen in $\binom{5}{r}$ ways, since $C(5, r) = \binom{5}{r}$. So the set has $\binom{5}{0}$ subsets with 0 elements, $\binom{5}{1}$ subsets with 1 element, $\binom{5}{2}$ subsets with 2 elements, and so on. Therefore, the total number of subsets is

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}. \tag{4}$$

Rather than calculate this sum directly, we can take advantage of the binomial theorem. Setting $n = 5$ gives

$$(x+y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5.$$

We want the right-hand side of this formula to equal the sum (5). This can be done by setting $x = 1$ and $y = 1$.

$$(1+1)^5 = \binom{5}{0}1^5 + \binom{5}{1}1^4 \cdot 1 + \binom{5}{2}1^3 \cdot 1^2 + \binom{5}{3}1^2 \cdot 1^3 + \binom{5}{4}1 \cdot 1^4 + \binom{5}{5}1^5$$
$$2^5 = \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$$

Thus, the total number of subsets of a set with five elements (the right side) equals $2^5 = 32$.

>> Now Try Exercise 37

There is nothing special about the number 5 in the preceding example. An analogous argument gives the following result:

> A set of n elements has 2^n subsets.

EXAMPLE 4 **Counting Pizza Options** A pizza parlor offers a plain cheese pizza to which any number of six possible toppings can be added. How many different pizzas can be ordered?

SOLUTION Ordering a pizza requires selecting a subset of the six possible toppings. Since the set of six toppings has 2^6 different subsets, there are 2^6, or 64, different pizzas. Note that the plain cheese pizza corresponds to selecting the empty subset of toppings.

>> Now Try Exercise 41

EXAMPLE 5 **Book Selection** In how many ways can a selection of at least two books be made from a set of six books?

SOLUTION There are $2^6 = 64$ different subsets of the six books. However, one subset contains no books and six subsets contain just one book. Therefore, the number of possibilities is $64 - 1 - 6 = 57$.

>> Now Try Exercise 43

Proof of the Binomial Theorem Note that

$$(x+y)^n = \underbrace{(x+y)(x+y) \cdot \cdots \cdot (x+y)}_{n \text{ factors}}.$$

Multiplying out these factors involves forming all products, where one term is selected from each factor, and then combining like products. For instance,

$$(x + y)(x + y)(x + y) = x \cdot x \cdot x + x \cdot x \cdot y + x \cdot y \cdot x + y \cdot x \cdot x + x \cdot y \cdot y + y \cdot x \cdot y + y \cdot y \cdot x + y \cdot y \cdot y.$$

The first product on the right, $x \cdot x \cdot x$, is obtained by selecting the x-term from each of the three factors. The next term, $x \cdot x \cdot y$, is obtained by selecting the x-terms from the first two factors and the y-term from the third. The next product, $x \cdot y \cdot x$, is obtained by selecting the x-terms from the first and third factors and the y-term from the second, and so on. There are as many products containing two x's and one y as there are ways of selecting the factor from which to pick the y-term—namely, $\binom{3}{1}$.

In general, when multiplying the n factors $(x + y)(x + y) \cdots (x + y)$, the number of products having k y's (and therefore, $(n - k)$ x's) is equal to the number of different ways of selecting the k factors from which to take the y-term—that is, $\binom{n}{k}$. Therefore, the coefficient of $x^{n-k}y^k$ is $\binom{n}{k}$. This proves the binomial theorem. «

Check Your Understanding 7

Solutions can be found following the section exercises.

1. Calculate $\binom{12}{8}$.

2. An ice cream parlor offers 10 flavors of ice cream and 5 toppings. How many different servings are possible if each serving consists of one flavor of ice cream and as many toppings as desired?

EXERCISES 7

Calculate the value for each of Exercises 1–18.

1. $C(18, 16)$
2. $C(25, 24)$
3. $\binom{6}{2}$
4. $\binom{7}{3}$
5. $\binom{8}{1}$
6. $\binom{9}{9}$
7. $\binom{7}{0}$
8. $\binom{6}{1}$
9. $\binom{8}{8}$
10. $\binom{9}{0}$
11. $\binom{n}{n-1}$
12. $\binom{n}{n}$
13. $0!$
14. $1!$
15. $n \cdot (n - 1)!$
16. $\frac{n!}{n}$
17. $\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$
18. $\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}$
19. How many terms are there in the binomial expansion of $(x + y)^{19}$?
20. How many terms are there in the binomial expansion of $(x + y)^{25}$?
21. Determine the first three terms in the binomial expansion of $(x + y)^{10}$.
22. Determine the first three terms in the binomial expansion of $(x + y)^{20}$.
23. Determine the last three terms in the binomial expansion of $(x + y)^{15}$.
24. Determine the last three terms in the binomial expansion of $(x + y)^{12}$.
25. Determine the middle term in the binomial expansion of $(x + y)^{20}$.
26. Determine the middle term in the binomial expansion of $(x + y)^{10}$.
27. Determine the coefficient of x^2 in the expansion of $(1 + x)^4$.
28. Determine the coefficient of x^3 in the expansion of $(2 + x)^6$.
29. Determine the coefficient of x^4y^7 in the binomial expansion of $(x + y)^{11}$.
30. Determine the coefficient of x^9y^4 in the binomial expansion of $(x + y)^{13}$.
31. Determine the first three terms in the binomial expansion of $(x + 2y)^9$.
32. Determine the last three terms in the binomial expansion of $(x - y)^8$.
33. Determine the middle term in the binomial expansion of $(x - 3y)^{12}$.
34. Determine the coefficient of x^4y^2 in the binomial expansion of $(x + 3y)^6$.
35. Determine the term containing y^3 in the expansion of $(x - 3y)^7$.
36. Determine the term containing y^5 in the expansion of $(x - 3y)^8$.
37. How many different subsets can be chosen from a set of eight elements?
38. How many different subsets can be chosen from a set of nine elements?
39. **Restaurant Tip** How many different tips could you leave in a tip jar if you had a nickel, a dime, a quarter, and a half-dollar?
40. **Pizza Options** A pizza parlor offers mushrooms, green peppers, onions, and sausage as toppings for the plain cheese base. How many different types of pizzas with no duplicate toppings can be made?
41. **Cable TV Options** A cable TV franchise offers 20 basic channels plus a selection (at an extra cost per channel) from a

collection of 5 premium channels. How many different options are available to the subscriber?

42. **Salad Options** A salad bar offers a base of lettuce to which tomatoes, chickpeas, beets, pinto beans, olives, and green peppers can be added. Five salad dressings are available. How many different salads are possible? (Assume that each salad contains at least lettuce and at most one salad dressing.)

43. **Tie Selection** In how many ways can a selection of at least one tie be made from a set of eight ties?

44. **Dessert Choices** In how many ways can a selection of at most five desserts be made from a dessert trolley containing six desserts?

45. **Pizza Options** Armand's Chicago Pizzeria offers thin-crust and deep-dish pizzas in 9-, 12-, and 14-inch sizes, with 13 possible toppings. How many different types of pizzas with no duplicate toppings can be ordered?

46. **Ice Cream Sundaes** An ice cream parlor offers four flavors of ice cream, three sauces, and two types of nuts. How many different sundaes consisting of a single flavor of ice cream plus one or more toppings are possible?

47. **Appetizer Selection** In how many ways can a selection of at most five appetizers be made from a menu containing seven appetizers?

48. **CD Selection** In how many ways can a selection of at least two CDs be made from a set of seven CDs?

49. **Lab Projects** Students in a physics class are required to complete at least two out of a collection of eight lab projects. In how many ways can a student satisfy the requirement?

50. **Election** A voter is asked to vote for at most six of the eight candidates for school board. In how many ways can the voter cast their ballot?

51. **CD Selection** James has nine jazz CDs and ten top 40/pop CDs. How many possible combinations of CDs can he select if he decides to take at least two CDs of each type to a party?

52. **Book Selection** Sarah has five nonfiction and six fiction books on her reading list. When packing for her summer vacation, she decides to pack at least two nonfiction books and at least one fiction book. How many different book selections are possible?

53. Can $\binom{8}{5}x^3y^4$ be a term of a binomial expansion?

54. Can $\binom{8}{5}x^3$ be a term of a binomial expansion?

55. Show that half of the subsets of a set of five elements have an odd number of elements. *Hint:* Show that $\binom{5}{0}+\binom{5}{2}+\binom{5}{4}=\binom{5}{1}+\binom{5}{3}+\binom{5}{5}$.

56. **(a)** Use equation (3) to show that

$$\binom{5}{0}-\binom{5}{1}+\binom{5}{2}-\binom{5}{3}+\binom{5}{4}-\binom{5}{5}=0.$$

(b) For what values of n can equation (3) be used to show that

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots\pm\binom{n}{n}=0?$$

(c) Use the binomial theorem to prove the result in part (a). *Hint:* Apply the binomial theorem to $(x+y)^5$ with $x=1$ and $y=-1$.

(d) For what values of n can the binomial theorem be used to prove the result in part (b)?

57. Find a simple expression for the value of

$$\binom{10}{1}+\binom{10}{2}+\binom{10}{3}+\binom{10}{4}+\binom{10}{5}+\binom{10}{6}+\binom{10}{7}+\binom{10}{8}+\binom{10}{9}.$$

58. **Seating Passengers** A blue van and a red van, each having nine passenger seats, have arrived to take ten people to the airport. In how many different ways can the passengers be placed into the vans?

Solutions to Check Your Understanding 7

1. 495. $\binom{12}{8}$ is the same as $C(12, 8)$, which equals $C(12, 12-8)$ or $C(12, 4)$.

$$C(12,4)=\frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}=\frac{\cancel{12}\cdot 11\cdot \overset{5}{\cancel{10}}\cdot 9}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}=495$$

2. 320. The task of deciding what sort of serving to have consists of two choices. The first choice, selecting the flavor of ice cream, can be performed in 10 ways. The second choice, selecting the toppings, can be performed in 2^5, or 32, ways, since selecting the toppings amounts to selecting a subset from the set of 5 toppings and a set of 5 elements has 2^5 subsets. (Notice that selecting the empty subset corresponds to ordering a plain dish of ice cream.) By the multiplication principle, the task can be performed in $10\cdot 32=320$ ways.

8 Multinomial Coefficients and Partitions

Permutation and combination problems are only two of the many types of counting problems. One such type, problems involving *partitions*, arise as a generalization of a problem we've already studied. To introduce the notion of a partition, let us return to combinations and look at them from another viewpoint.

First, consider the number of ways to select two elements from the set $\{a, b, c\}$:

$$\{a, b\} \qquad \{a, c\} \qquad \{b, c\}$$

Alternatively, we can write the selection $\{a, b\}$ as $(\{a, b\}, \{c\})$, separating the elements of the set $\{a, b, c\}$ that we selected from the element we did not select. We call $(\{a, b\}, \{c\})$ an **ordered partition** of type (2, 1). There are three such partitions of the set:

$$(\{a, b\}, \{c\}), \qquad (\{a, c\}, \{b\}) \qquad (\{b, c\}, \{a\})$$

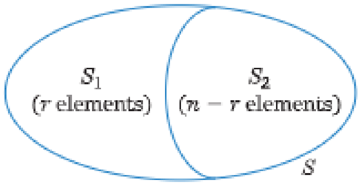


Figure 1

In general, suppose that we consider combinations of n objects taken r at a time. View the n objects as the elements of a set S. Then each combination determines an ordered division of S into two subsets, S_1 and S_2, the first containing the r elements selected and the second containing the $n - r$ elements remaining (Fig. 1). We see that

$$S = S_1 \cup S_2 \quad \text{and} \quad n(S_1) + n(S_2) = n.$$

This ordered division is called an **ordered partition of type $(r, n - r)$**. We know that the number of such partitions is just the number of ways of selecting the first subset—that is, $n!/[r!(n-r)!]$. If we let $n_1 = n(S_1) = r$ and $n_2 = n(S_2) = n - r$, then we find that the number of ordered partitions of type (n_1, n_2) is $n!/[n_1!\, n_2!]$.

We may generalize the aforementioned situation as follows: Let S be a set of n elements. An **ordered partition of S of type $(n_1, n_2, \ldots, n_m)$** is a decomposition of S into m subsets (given in a specific order) $S_1, S_2, \ldots, S_m$, where no two of these intersect and where

$$n(S_1) = n_1, \qquad n(S_2) = n_2, \quad \ldots \quad , n(S_m) = n_m$$

(Fig. 2). Since S has n elements, we clearly must have $n = n_1 + n_2 + \cdots + n_m$.

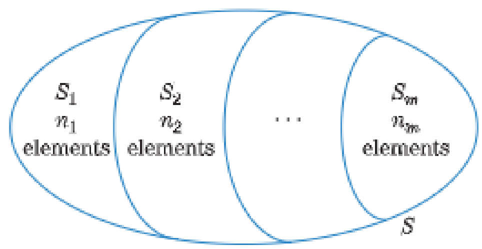


Figure 2

EXAMPLE 1 **Ordered Partitions of a Set** List all ordered partitions of $S = \{a, b, c, d\}$ of type (1, 1, 2).

SOLUTION

$$\begin{array}{ll}
(\{a\}, \{b\}, \{c, d\}) & (\{c\}, \{a\}, \{b, d\}) \\
(\{a\}, \{c\}, \{b, d\}) & (\{c\}, \{b\}, \{a, d\}) \\
(\{a\}, \{d\}, \{b, c\}) & (\{c\}, \{d\}, \{a, b\}) \\
(\{b\}, \{a\}, \{c, d\}) & (\{d\}, \{a\}, \{b, c\}) \\
(\{b\}, \{c\}, \{a, d\}) & (\{d\}, \{b\}, \{a, c\}) \\
(\{b\}, \{d\}, \{a, c\}) & (\{d\}, \{c\}, \{a, b\})
\end{array}$$

Note that the ordered partition $(\{a\},\{b\},\{c, d\})$ is different from the ordered partition $(\{b\},\{a\},\{c, d\})$, since in the first, S_1 is $\{a\}$, whereas in the second, S_1 is $\{b\}$. The order in which the subsets are given is significant. «

We saw earlier that the number of ordered partitions of type (n_1, n_2) for a set of n elements is $n!/[n_1!\, n_2!]$. This result generalizes.

Number of Ordered Partitions of Type $(n_1, n_2, \ldots, n_m)$ Let S be a set of n elements. Then the number of ordered partitions of S of type $(n_1, n_2, \ldots, n_m)$, where $n_1 + n_2 + \cdots + n_m = n$ is

$$\frac{n!}{n_1!\, n_2! \cdots n_m!}. \qquad (1)$$

The number of ordered partitions of type $(n_1, n_2, \ldots, n_m)$ for a set of n elements is often denoted

$$\binom{n}{n_1, n_2, \ldots, n_m}.$$

Using the preceding notation, result (1) says that

$$\binom{n}{n_1, n_2, \ldots, n_m} = \frac{n!}{n_1!\, n_2! \cdots n_m!}.$$

The binomial coefficient $\binom{n}{r}$ can also be written $\binom{n}{r, n-r}$. The number

$$\binom{n}{n_1, n_2, \ldots, n_m}$$

is known as a **multinomial coefficient**, since it appears as the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$ in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$.

EXAMPLE 2 **Ordered Partitions of a Set** Let S be a set of four elements. Use the formula in (1) to determine the number of ordered partitions of S of type (1, 1, 2).

SOLUTION Here, $n = 4$, $n_1 = 1$, $n_2 = 1$, and $n_3 = 2$. Therefore, the number of ordered partitions of type (1, 1, 2) is

$$\binom{4}{1, 1, 2} = \frac{4!}{1!\, 1!\, 2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 1 \cdot 2 \cdot 1} = 12.$$

This result is the same as obtained in Example 1 by enumeration.

>> Now Try Exercise 1

EXAMPLE 3 **Assigning Tasks to Construction Workers** A work crew consists of 12 construction workers, all having the same skills. A construction job requires four welders, three concrete workers, three heavy equipment operators, and two bricklayers. In how many ways can the 12 workers be assigned to the required tasks?

SOLUTION Each assignment of jobs corresponds to an ordered partition of the type (4, 3, 3, 2). The number of such ordered partitions is

$$\binom{12}{4, 3, 3, 2} = \frac{12!}{4!\, 3!\, 3!\, 2!} = 277{,}200.$$

>> Now Try Exercise 15

Sometimes, the m subsets of an ordered partition are required to have the same number of elements. If the set has n elements and each of the m subsets has r elements, then the number of ordered partitions of type

$$(\underbrace{r, r, \ldots, r}_{m})$$

is

$$\binom{n}{r, r, \ldots, r} = \frac{n!}{r!\, r! \cdots r!} = \frac{n!}{(r!)^m}. \tag{2}$$

EXAMPLE 4 **Counting the Number of Bridge Hands** In the game of bridge, four players seated in a specific order are each dealt 13 cards. How many different possibilities are there for the hands dealt to the players?

SOLUTION Each deal results in an ordered partition of the 52 cards of type (13, 13, 13, 13). The number of such partitions is

$$\binom{52}{13, 13, 13, 13} = \frac{52!}{(13!)^4}.$$

This number is approximately 5.36×10^{28}. «

EXAMPLE 5 **MISSISSIPPI** How many 11-letter words consist of one M, four Is, four Ss, and two Ps?

SOLUTION Think of forming a word as filling slots numbered 1, 2, . . . , 11 with the letters. Partition the set of 11 slots into four subsets where the first subset gives the location for the M, the second subset gives the locations for the four Is, the third subset gives the locations for the four Ss, and the fourth subset gives the locations for the two Ps. The number of partitions is

$$\binom{11}{1, 4, 4, 2} = \frac{11!}{1!\,4!\,4!\,2!} = \frac{39{,}916{,}800}{1 \cdot 24 \cdot 24 \cdot 2} = \frac{39{,}916{,}800}{1152} = 34{,}650.$$

›› *Now Try Exercise 17*

Unordered Partitions

Determining the number of unordered partitions of a certain type is a complex matter. We will restrict our attention to the special case in which each subset is of the same size.

EXAMPLE 6 **Unordered Partitions of a Set** List all unordered partitions of $S = \{a, b, c, d\}$ of type (2, 2).

SOLUTION

$$(\{a, b\}, \{c, d\})$$
$$(\{a, c\}, \{b, d\})$$
$$(\{a, d\}, \{b, c\})$$ «

Note that the partition $(\{c, d\}, \{a, b\})$ is the same as the partition $(\{a, b\}, \{c, d\})$ when order is not taken into account.

Number of Unordered Partitions of Type $(r, r, \ldots, r)$ Let S be a set of n elements where $n = m \cdot r$. Then the number of unordered partitions of S of type $(r, r, \ldots, r)$ is

$$\frac{1}{m!} \cdot \frac{n!}{(r!)^m}. \qquad (3)$$

Formula (3) follows from the fact that each unordered partition of the m subsets gives rise to $m!$ ordered partitions. Therefore,

$$(m!)\,[\text{number of unordered partitions}] = [\text{number of ordered partitions}]$$

or

$$[\text{number of unordered partitions}] = \frac{1}{m!} \cdot [\text{number of ordered partitions}]$$
$$= \frac{1}{m!} \cdot \frac{n!}{(r!)^m} \quad [\text{by formula (2)}].$$

EXAMPLE 7 **Unordered Partitions of a Set** Let S be a set of four elements. Use formula (3) to determine the number of unordered partitions of S of type (2, 2).

SOLUTION Here, $n = 4$, $r = 2$, and $m = 2$. Therefore, the number of unordered partitions of type (2, 2) is

$$\frac{1}{2!} \cdot \frac{4!}{(2!)^2} = \frac{1}{2} \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)^2} = 3.$$

This result is the same as that obtained in Example 6 by enumeration.

>> Now Try Exercise 11

EXAMPLE 8 **Grouping Construction Workers** A construction crew contains 12 workers, all having similar skills. In how many ways can the workers be divided into four groups of three?

SOLUTION The order of the four groups is not relevant. (It does not matter which is labeled S_1 and which S_2, and so on. Only the composition of the groups is important.) Applying formula (3) with $n = 12$, $r = 3$, and $m = 4$, we see that the number of ways is

$$\frac{1}{m!} \cdot \frac{n!}{(r!)^m} = \frac{1}{4!} \cdot \frac{12!}{(3!)^4}$$
$$= 15{,}400.$$

>> Now Try Exercise 21

INCORPORATING TECHNOLOGY

WolframAlpha The number of ordered partitions of type $(n_1, n_2, \ldots, n_m)$ is given by the instruction

multinomial $(n_1, n_2, \ldots, n_m)$.

For instance, the answer to Example 3 is given by **multinomial (4, 3, 3, 2)** and the answer to Example 8 is given by **(1/4!)*multinomial (3, 3, 3, 3)**.

Check Your Understanding 8

Solutions can be found following the section exercises.

1. A foundation wishes to award one grant of \$100,000, two grants of \$10,000, five grants of \$5000, and five grants of \$2000. Its list of potential grant recipients has been narrowed to 13 possibilities. In how many ways can the awards be made?

2. In how many different ways can six medical interns be put into three groups of two and assigned to
   (a) the radiology, neurology, and surgery departments?
   (b) share offices?

EXERCISES 8

Let S be a set of n elements. Determine the number of ordered partitions of the types in Exercises 1–10.

1. $n = 5$; (3, 1, 1)
2. $n = 5$; (2, 1, 2)
3. $n = 6$; (2, 1, 2, 1)
4. $n = 6$; (3, 3)
5. $n = 7$; (3, 2, 2)
6. $n = 7$; (4, 1, 2)
7. $n = 12$; (4, 4, 4)
8. $n = 8$; (3, 3, 2)
9. $n = 12$; (5, 3, 2, 2)
10. $n = 8$; (2, 2, 2, 2)

Let S be a set of n elements. Determine the number of unordered partitions of the types in Exercises 11–14.

11. $n = 15$; (3, 3, 3, 3, 3)
12. $n = 10$; (5, 5)
13. $n = 18$; (6, 6, 6)
14. $n = 12$; (4, 4, 4)

15. **Stock Reports** A brokerage house regularly reports the behavior of a group of 20 stocks, each stock being reported as "up," "down," or "unchanged." How many different reports can show seven stocks up, five stocks down, and eight stocks unchanged?

16. **Investment Ratings** An investment advisory service rates investments as A, AA, and AAA. On a certain week, it rates 15 investments. In how many ways can it rate five investments in each of the categories?

17. **RIFFRAFF** How many eight-letter words consist of two Rs, one I, four Fs, and one A?

18. **RAZZMATAZZ** How many 10-letter words consist of one R, three As, four Zs, one M, and one T?

19. **Postage Stamps** In how many ways can three Forever stamps, two 20¢ stamps, and four 1¢ stamps be pasted in a row on an envelope?

20. **Nautical Signals** A nautical signal consists of six flags arranged vertically on a flagpole. If a sailor has three red flags, two blue flags, and one white flag, how many different signals are possible?

21. **Observation Groups** A psychology experiment observes groups of four individuals. In how many ways can an experimenter choose 5 groups of 4 from among 20 subjects?

22. **Orientation Groups** During orientation, new students are divided into groups of five people. In how many ways can 4 groups be chosen from among 20 people?

23. **Weather** In a certain month (of 30 days) it rains 10 days, snows 2 days, and is clear 18 days. In how many ways can such weather be distributed over the month?

24. **Awarding Prizes** Of the nine contestants in a contest, three will receive cars, three will receive TV sets, and three will receive radios. In how many different ways can the prizes be awarded?

25. **Job Promotions** A corporation has four employees that it wants to place in high executive positions. One will become president, one will become vice president, and two will be appointed to the board of directors. In how many different ways can this be accomplished?

26. **Forming Committees** The 10 members of a city council decide to form two committees of six to study zoning ordinances and street-repair schedules, with an overlap of two committee members. In how many ways can the committees be formed? (*Hint:* Specify three groups, not two.)

27. **Field Trip** In how many ways can the 14 children in a third-grade class be paired up for a trip to a museum?

28. **Work Schedule** A sales representative must travel to three cities, twice each, in the next 10 days. Her nontravel days are spent in the office. In how many different ways can she schedule her travel?

29. **Basketball Teams** Ten students in a physical education class are to be divided into five-member teams for a basketball game. In how many ways can the two teams be selected?

30. **Sorting Sweaters** In how many ways can 12 sweaters be stored into three boxes of different sizes if 6 sweaters are to be stored in the large box, 4 in the medium box, and 2 in the small box?

31. What is the value of $\binom{n}{1,1,\ldots,1}$ where there are n 1s?

32. Derive formula (1), using the generalized multiplication principle and the formula for $\binom{n}{r}$. (*Hint:* First select the elements of S_1, then the elements of S_2, and so on.)

TECHNOLOGY EXERCISES

33. **Assignments to Seminars** Calculate the number of ways that 38 students can be assigned to four seminars of size 10, 12, 10, and 6, respectively.

34. **Campaign Tasks** Calculate the number of ways that 65 phone numbers can be distributed to 5 campaign workers if each worker gets the same number of names.

35. **Bridge Deals** One octillion is 10^{28}, or 10 billion billion billion. Is the number of possible deals in bridge greater than or less than one octillion?

Solutions to Check Your Understanding 8

1. Each choice of recipients is an ordered partition of the 13 finalists into a first subset of one ($100,000 award), a second subset of two ($10,000 award), a third subset of five ($5000 award), and a fourth subset of five ($2000 award). The number of ways to choose the recipients is thus

$$\binom{13}{1,2,5,5} = \frac{13!}{1!\,2!\,5!\,5!}$$

$$= \frac{13\cdot12\cdot11\cdot\cancel{10}\cdot\overset{3}{\cancel{9}}\cdot\cancel{8}\cdot7\cdot6\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{1\cdot\cancel{2}\cdot1\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$= 13\cdot12\cdot11\cdot3\cdot7\cdot6 = 216{,}216.$$

2. Each partition is of the type (2, 2, 2). In part (a), the order of the subsets is important, whereas in part (b), the order is irrelevant. Consider the partitions

({Dr. A, Dr. B}, {Dr. C, Dr. D}, {Dr. E, Dr. F})

and

({Dr. C, Dr. D}, {Dr. A, Dr. B}, {Dr. E, Dr. F}).

With respect to part (a), these two partitions are different, since in one, Drs. A and B are assigned to the radiology department and in the other, they are assigned to the neurology department. With respect to part (b), these two partitions are the same, since, for instance, Drs. A and B are officemates in both partitions. Therefore, the answers are as follows:

(a) $\binom{6}{2,2,2} = \frac{6!}{(2!)^3} = \frac{6\cdot5\cdot\cancel{4}\cdot3\cdot\cancel{2}\cdot1}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} = 90.$

(b) $\frac{1}{3!}\cdot\frac{6!}{(2!)^3} = \frac{1}{\cancel{6}}\cdot\frac{\cancel{6}\cdot5\cdot\cancel{4}\cdot3\cdot\cancel{2}\cdot1}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} = 15.$

Summary

KEY TERMS AND CONCEPTS / EXAMPLES

1 Sets

Let A and B be sets with universal set U.

Let $U = \{a, b, c, d, e, f\}$, $A = \{a, b, c, d\}$, and $B = \{c, d, f\}$. Then,

The **intersection** of A and B, written $A \cap B$, is the set of all elements that belong to both A and B.

$$A \cap B = \{c, d\}$$

The **union** of A and B, written $A \cup B$, is the set of all elements that belong to either A or B or both.

$$A \cup B = \{a, b, c, d, f\}$$

The **complement** of A, written A', is the set of all elements in U that do not belong to A.

$$A' = \{e, f\}.$$

The set S is a **subset** of the set T, written $S \subseteq T$, if every element of S is an element of T.

$$\{a, c, f\} \subseteq \{a, b, c, d, e, f\}$$

The **empty set**, denoted Ø, is the set containing no elements.

2 A Fundamental Principle of Counting

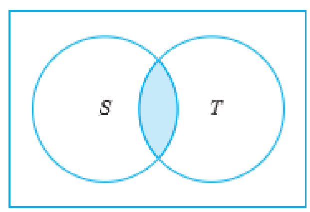
Number of elements in set S is denoted $\boldsymbol{n(S)}$.

$n(\{a, b, c\}) = 3$

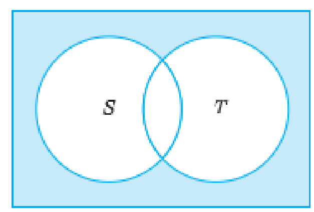
Inclusion–Exclusion Principle:
$n(S \cup T) = n(S) + n(T) - n(S \cap T)$.

Let $S = \{a, b, c, d\}$ and $T = \{b, d, e\}$. Then,
$n(S) = 4$, $n(T) = 3$, $n(S \cap T) = 2$, and $S \cup T = \{a, b, c, d, e\}$.
By the inclusion–exclusion principle, $n(S \cup T) = 4 + 3 - 2 = 5$.

Venn diagrams are used to illustrate set operations.



$S \cap T$



$(S \cup T)'$

De Morgan's Laws:
$(S \cup T)' = S' \cap T'$ and $(S \cap T)' = S' \cup T'$.

Let $U = \{a, b, c, d, e, f\}$, $S = \{a, b, c, d\}$, and $T = \{c, d, f\}$. Then,

$$(S \cup T)' = \{a, b, c, d, f\}' = \{e\}$$
$$S' \cap T' = \{e, f\} \cap \{a, b, e\} = \{e\}$$
$$(S \cap T)' = \{c, d\}' = \{a, b, e, f\}$$
$$S' \cup T' = \{e, f\} \cup \{a, b, e\} = \{a, b, e, f\}.$$

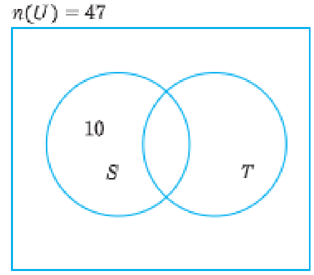
3 Venn Diagrams and Counting

Venn diagrams can be used to solve certain types of counting problems.

Suppose that $n(U) = 47$, $n(S \cap T') = 10$, $n(S) = 15$, $n(T) = 25$.
Find $n((S \cup T)')$.

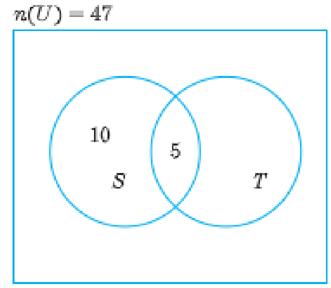
$n(U) = 47$
$n(S \cap T') = 10$

$n(U) = 47$

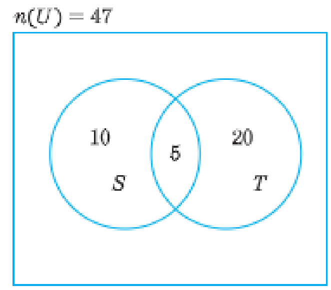


KEY TERMS AND CONCEPTS	EXAMPLES

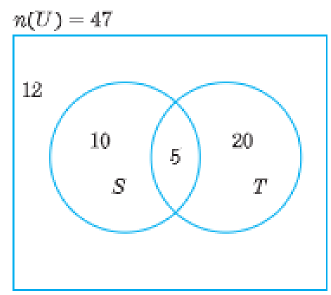
Since $n(S) = 15$, $n(S \cap T) = 5$



Since $n(T) = 25$, $n(S' \cap T) = 20$



$n((S \cup T)') =$
$47 - 10 - 5 - 20 = 12$



4 The Multiplication Principle

The **multiplication principle** states that if there are m ways to make a selection and, for each of these, there are n ways to make a second selection, then there are $m \times n$ ways to make the two selections.

If you have three coats and four hats, then there are $3 \times 4 = 12$ ways to select a coat and hat.

The multiplication principle generalizes to any number of selections.

If you also have two scarves, there are $3 \times 4 \times 2 = 24$ ways to select a coat, hat, and scarf.

5 Permutations and Combinations

The number of **permutations** of n elements taken r at a time (ordered selections) is

$$P(n, r) = \frac{n!}{(n-r)!} = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ factors}}.$$

How many ways can you select three of ten photos and hang them in a row on the wall?

Answer: $P(10, 3) = 10 \cdot 9 \cdot 8 = 720$

The number of **combinations** of n elements taken r at a time (unordered selections) is

$$C(n, r) = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1}.$$

How many ways can you select three of ten photos?

Answer: $C(10, 3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$

6 Further Counting Techniques

The **complement rule of counting** says that $n(S) = n(U) - n(S')$.

When a coin is tossed five times, how many possible outcomes contain two or more heads?

Answer: Let S be the set of outcomes containing two or more heads. Then S' is the number of outcomes containing zero or one head. Since $n(S') = 1 + 5 = 6$, $n(S) = 2^5 - 6 = 32 - 6 = 26$.

KEY TERMS AND CONCEPTS	EXAMPLES
7 The Binomial Theorem	
$C(n, r) = C(n, n - r)$	$C(5, 3) = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = \frac{5 \cdot 4 \cdot \not{3}}{\not{3} \cdot 2 \cdot 1} = C(5, 2)$
The **binomial theorem** states that $(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$	$(x + y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^{3-1}y + \binom{3}{2}x^{3-2}y^2 + \binom{3}{3}x^{3-3}y^3 = x^3 + 3x^2y + 3xy^2 + y^3$
A set of n elements has 2^n subsets.	The set $\{a, b, c\}$ has 3 elements and $2^3 = 8$ subsets: $\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \varnothing$.
8 Multinomial Coefficients and Partitions	
The number of **ordered partitions** of type $(n_1, n_2, \ldots, n_m)$ for a set of n elements is $\frac{n!}{n_1!n_2!\ldots n_m!}$, denoted $\binom{n}{n_1, n_2, \ldots, n_m}$.	How many different signals can be created by lining up 10 flags in a vertical column if 2 flags are green, 3 are red, and 5 are blue? A signal is an ordered partition of the numbers 1 through 10. For instance, one example of an ordered partition is ({3, 7}, {1, 4, 9}, {2, 5, 6, 8, 10}) corresponding to the green flags being in the 3rd and 7th positions, the red flags being in the 1st, 4th, and 9th positions, and the blue flags being in the remaining positions. Therefore, the total number of signals is $\frac{10!}{2!3!5!} = 2520$.
The number of **unordered partitions** of type $(r, r, \ldots, r)$ for a set of n elements where $n = m \cdot r$ is $\frac{1}{m!} \cdot \frac{n!}{(r!)^m}$.	In how many ways can six students be paired up to work on a group project? Answer: Since $n = 6$, $r = 2$, $m = 3$, there are $\frac{1}{3!} \cdot \frac{6!}{(2!)^3} = 15$ different ways.

Fundamental Concept Check Exercises

1. What is a set?
2. What is a subset of a set?
3. What is an element of a set?
4. Define a universal set.
5. Define the empty set.
6. Define the complement of the set A.
7. Define the intersection of the two sets A and B.
8. Define the union of the two sets A and B.
9. State the generalized multiplication principle for counting.
10. What is meant by a permutation of n items taken r at a time?
11. How would you calculate the number of permutations of n items taken r at a time?
12. What is the difference between a permutation and a combination?
13. How would you calculate the number of combinations of n items taken r at a time?
14. Give a formula that can be used to calculate each of the following:

$$n! \qquad \binom{n}{r} \qquad C(n, r) \qquad P(n, r)$$

15. State the binomial theorem.
16. If a set contains n elements, how many subsets does it have?
17. Explain what is meant by an ordered partition of a set.
18. Explain how to calculate the number of ordered partitions of a set.

Review Exercises

1. List all subsets of the set $\{a, b\}$.
2. Draw a two-circle Venn diagram, and shade the portion corresponding to the set $(S \cup T')'$.
3. **Tennis Finalists** There are 16 contestants in a tennis tournament. How many different possibilities are there for the two people who will play in the final round?
4. **Team Picture** In how many ways can a coach and five basketball players line up in a row for a picture if the coach insists on standing at one of the ends of the row?
5. Draw a three-circle Venn diagram, and shade the portion corresponding to the set $R' \cap (S \cup T)$.

6. Determine the first three terms in the binomial expansion of $(x - 2y)^{12}$.

7. **Balls in an Urn** An urn contains 14 numbered balls, of which 8 are red and 6 are green. How many different possibilities are there for selecting a sample of 5 balls in which 3 are red and 2 are green?

8. **Testing a Drug** Sixty people with a certain medical condition were given pills. Fifteen of these people received placebos. Forty people showed improvement, and 30 of these had received an actual drug. How many of the people who received the drug showed no improvement?

9. **Appliance Purchase** An appliance store carries seven different types of washing machines and five different types of dryers. How many different combinations are possible for a customer who wants to purchase a washing machine and a dryer?

10. **Contest Prizes** There are 12 contestants in a contest. Two will receive trips around the world, four will receive cars, and six will receive TV sets. In how many different ways can the prizes be awarded?

11. **Languages** Out of a group of 115 applicants for jobs at the World Bank, 70 speak French, 65 speak Spanish, 65 speak German, 45 speak French and Spanish, 35 speak Spanish and German, 40 speak French and German, and 35 speak all three languages. How many of the people speak none of the three languages?

12. Calculate $\binom{17}{15}$.

Environmental Poll The 100 members of the Earth Club were asked what they felt the club's priorities should be in the coming year: clean water, clean air, or recycling. The responses were 45 for clean water, 30 for clean air, 42 for recycling, 13 for both clean air and clean water, 20 for clean air and recycling, 16 for clean water and recycling, and 9 for all three. Exercises 13–20 refer to this poll.

13. How many members thought that the priority should be clean air only?

14. How many members thought that the priority should be clean water or clean air, but not both?

15. How many members thought that the priority should be clean water or recycling but not clean air?

16. How many members thought that the priority should be clean air and recycling but not clean water?

17. How many members thought that the priority should be exactly one of the three issues?

18. How many members thought that recycling should not be a priority?

19. How many members thought that the priority should be recycling but not clean air?

20. How many members thought that the priority should be something other than one of these three issues?

21. **Nine-Letter Words** How many different nine-letter words (i.e., sequences of letters) can be made by using four Ss and five Ts?

22. **Passing an Exam** Twenty people take an exam. How many different possibilities are there for the set of people who pass the exam?

23. **Winter Sports** A survey at a small New England college showed that 400 students skied, 300 played ice hockey, and 150 did both. How many students participated in at least one of these sports?

24. **Meal Choices** How many different meals can be chosen if there are 6 appetizers, 10 main dishes, and 8 desserts, assuming that a meal consists of one item from each category?

25. **Test Scoring** On an essay test, there are five questions worth 20 points each. In how many ways can a student get 10 points on one question, 15 points on each of three questions, and 20 points on another question?

26. **Seven-Digit Numbers** How many seven-digit numbers are even and have a 3 in the hundreds place?

27. **Telephone Numbers** How many telephone numbers are theoretically possible if all numbers are of the form *abc-def-ghij* and neither of the first two leading digits (a and d) is zero?

28. **Three-Digit Numbers** How many three-digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6, and 7 if no digit is repeated?

29. **Eight-Letter Words** How many strings of length 8 can be formed from the letters A, B, C, D, and E ? How many of the strings have at least one E?

30. **Basketball Teams** How many different five-person basketball teams can be formed from a pool of 12 players?

31. **Selecting Students** Fourteen students in the 30-student eighth grade are to be chosen to tour the United Nations. How many different groups of 14 are possible?

32. **Journal Subscriptions** In one ZIP code, there are 40,000 households. Of them, 4000 households get *Fancy Diet Magazine*, 10,000 households get *Clean Living Journal*, and 1500 households get both publications. How many households get neither?

33. **Computer Program** At each stage in a decision process, a computer program has three branches. There are 10 stages at which these branches appear. How many different paths could the process follow?

34. **Filling Jobs** Sixty people apply for 10 job openings. In how many ways can all of the jobs be filled?

35. **Generating Tests** A computerized test generator can generate any one of five problems for each of the 10 areas being tested. How many different tests can be generated?

36. **Six-Letter Words** If a string of six letters cannot contain any vowels (A, E, I, O, U), how many strings are possible?

37. **Choosing Delegations** How many different four-person delegations can be chosen from 10 ambassadors?

38. **Subdividing a Class** In how many ways can a teacher divide a class of 21 students into groups of 7 students each?

39. **Distributing Candy** In how many ways can 14 different candies be distributed to 14 scouts?

40. **Subdividing Peo ple** In how many ways can 20 people be divided into groups of 5 each?

41. **Executive Positions** An Internet company is considering three candidates for CEO, five candidates for CFO, and four candidates for marketing director. In how many different ways can these positions be filled?

42. **Softball League** If every team in a 10-team softball league plays every other team three times, how many games are played?

43. **Diagonals** How many diagonals does an n-sided polygon have?

44. **Racetrack Betting** Racetracks have a compound bet called the *daily double*, in which the bettor tries to select the winners of the first two races. If eight horses compete in the first race and six horses compete in the second race, determine the possible number of daily double bets.

45. **Hat Displays** A designer of a window display wants to form a pyramid with 15 hats. She wants to place the five men's hats in the bottom row, the four women's hats in the next row, next the three baseball caps, then the two berets, and a clown's hat at the top. All of the hats are different. How many displays are possible?

46. **Seat Assignments** In how many ways can five people be assigned to seats in a 12-seat room?

47. **Poker** A poker hand consists of five cards. How many different poker hands contain all cards of the same suit? (Such a hand is called a "flush.")

48. **Poker** How many hands of five cards contain exactly three aces?

49. **Three-Digit Numbers** How many three-digit numbers are there in which exactly two digits are alike?

50. **Three-Digit Numbers** How many three-digit numbers are there in which no two digits are alike?

Baseball In Exercises 51 and 52, suppose there are three pitchers and three catchers at a baseball training camp.

51. How many different pitcher–catcher pairs can be formed?

52. In how many ways can the six players be seated in a row if no one sits next to someone who plays the same position?

53. **Greek-Letter Societies** Fraternity and sorority names consist of two or three letters from the Greek alphabet. How many different names are there in which no letter appears more than once? (The Greek alphabet contains 24 letters.)

54. **Family Picture** A family consisting of two parents and four children is to be seated in a row for a picture. How many different arrangements are possible in which the children are seated together?

55. **Feasible Set** If 10 lines are drawn in the plane so that no two of them are parallel and no three lines intersect at the same point, how many points of intersection are there?

56. **Splitting Classes** Two elementary school teachers have 24 students each. The first teacher splits his students into four groups of six. The second teacher splits her students into six groups of four. Which teacher has more options?

57. **Consultation Schedule** A consulting engineer agrees to spend three days at Widgets International, four days at Gadgets Unlimited, and three days at Doodads Incorporated in the next two workweeks. In how many different ways can she schedule her consultations?

58. **Arranging Books** A set of books can be arranged on a bookshelf in 120 different ways. How many books are in the set?

59. **Batting Order** How many batting orders are possible in a nine-member baseball team if the catcher must bat fourth and the pitcher last?

60. **Choosing Committees** In the United States Senate, each state is represented by one junior senator and one senior senator. In how many ways can a committee of five senators from the 12 Midwest states be represented if
(a) the committee must consist of two senior senators and three junior senators?
(b) no two senators from the same state can serve on the committee?

61. **Voting** Suppose that you are voting in an election for state delegate. Two state delegates are to be elected from among seven candidates. In how many different ways can you cast your ballot? *Note:* You may vote for two candidates. However, some people "single-shoot," and others don't pull any levers.

62. **Spelling *Algebra*** The object is to start with the letter A on top and to move down the diagram to an A at the bottom. From any given letter, move only to one of the letters directly below it on the left or right. If these rules are followed, how many different paths spell *ALGEBRA*?

A
L L
G G G
E E E E
B B B B B
R R R R R R
A A A A A A A

63. **Call Letters** The call letters of radio stations in the United States consist of either three or four letters, where the first letter is K or W. How many different call letters are possible?

TECHNOLOGY EXERCISES

In Exercises 64–66, use technology to calculate the answer.

64. **Arranging Students** A group of students can be arranged in a row of seats in 479,001,600 ways. How many students are there?

65. **Assigning Jobs** There are 25 people in a department who must be deployed to work on three projects requiring 10, 9, and 6 people. All of the people are eligible for all jobs. Calculate the number of ways this can be done.

66. **License Plates** Calculate the number of license plates that can be formed by using three distinct letters and three distinct digits in any order.

Conceptual Exercises

67. What is the relationship between the two sets A and B if $A \cap B = \varnothing$?

68. What is the relationship between the two sets A and B if $n(A \cup B) = n(A) + n(B)$?

69. Suppose that A and B are subsets of the set U. Under what circumstance will $A \cap B = B$?

70. Suppose that A and B are subsets of the set U. Under what circumstance will $A \cup B = B$?

71. (True or False) $n(A) + n(B) = n(A \cap B) + n(A \cup B)$.

72. (True or False) The empty set is a subset of every set.

73. Explain why $n \cdot (n-1)! = n!$.

74. Use the result in Exercise 73 to explain why 0! is defined to be 1.

75. Express in your own words the difference between a permutation and a combination.

76. Consider a group of 10 people. Without doing any computation, explain why the number of committees of six people is equal to the number of committees of four people.

77. Without doing any computation, explain why $C(10, 3) = C(10, 7)$.

78. Without doing any computation, explain why $C(10, 4) + C(10, 5) = C(11, 5)$. *Hint:* Suppose that a committee of size five is to be chosen from a pool of 11 people, and John Doe is one of the people. How many committees are there that include John? How many committees are there that don't include John?

CHAPTER

PROJECT

Pascal's Triangle

In the following triangular table, known as **Pascal's triangle**, the entries in the nth row are the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$.

1	0th row
1 1	1st row
1 2 1	2nd row
1 3 3 1	3rd row
1 4 6 4 1	4th row
1 5 10 10 5 1	5th row
1 6 15 20 15 6 1	6th row
1 7 21 35 35 21 7 1	7th row

Observe that each number (other than the ones) is the sum of the two numbers directly above it. For example, in the 5th row, the number 5 is the sum of the numbers 1 and 4 from the 4th row, and the number 10 is the sum of the numbers 4 and 6 from the 4th row. This fact is known as **Pascal's formula**. Namely, the formula says that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

1. For what values of n and r does Pascal's formula say that the number 10 in the triangle is the sum of the numbers 4 and 6?

2. Derive Pascal's formula from the fact that $C(n, r) = \dfrac{n!}{r!\,(n-r)!}$.

3. Derive Pascal's formula from the fact that $C(n, r)$ is the number of ways of selecting r objects from a set of n objects. (*Hint:* Let x denote the nth object of the set. Count the number of ways that a subset of r objects containing x can be selected, and then count the number of ways that a subset of r objects not containing x can be selected.)

4. Use Pascal's formula to extend Pascal's triangle to the 12th row. Determine the values of $\binom{12}{5}$ and $\binom{12}{6}$ from the extended triangle.

5. (a) Show that, for any positive integer n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n.$$

Hint: Apply the binomial theorem to $(x + y)^n$ with $x = 1, y = 1$.

(b) Show that, for any positive integer n,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = 0.$$

Hint: Apply the binomial theorem to $(x + y)^n$ with $x = 1, y = -1$.

(c) Show that, for the 7th row of Pascal's triangle, the sum of the even-numbered elements equals the sum of the odd-numbered elements; that is,

$$\binom{7}{0}+\binom{7}{2}+\binom{7}{4}+\binom{7}{6}=\binom{7}{1}+\binom{7}{3}+\binom{7}{5}+\binom{7}{7}.$$

(d) Use the result of parts (a) and (b) to show that, for any row of Pascal's triangle, the sum of the even-numbered elements equals the sum of the odd-numbered elements, and give that common sum for the nth row in terms of n. [*Hint:* Add the two equations in parts (a) and (b).]

(e) Suppose that S is a set of n elements. Use the result of part (d) to determine the number of subsets of S that have an even number of elements.

6. (a) Show that, for any positive integer n,

$$1+2+4+8+\cdots+2^n=2^{n+1}-1.$$

(*Hint:* Let $S=1+2+4+8+\cdots+2^n$, multiply both sides of the equation by 2, and subtract the first equation from the second.)

(b) Show that the sum of the elements of any row of Pascal's triangle equals one more than the sum of the elements of all previous rows.

7. (a) Consider the 7th row of Pascal's triangle. Observe that each interior number (that is, a number other than 1) is divisible by 7. For what values of n, for $1 \le n \le 12$, are the interior numbers of the nth row divisible by n?

(b) Confirm that each of the values of n from part (a) is a prime number. (*Note:* A number p is a *prime* number if the only positive integers that divide it are p and 1.) Prove that, if p is a prime number, then each interior number of the pth row of Pascal's triangle is divisible by p. $\left[\textit{Hint:}\text{ Use the fact that } \binom{p}{r}=\frac{p(p-1)(p-2)\cdots(p-r+1)}{1\cdot 2\cdot 3\cdot\cdots\cdot r}.\right]$

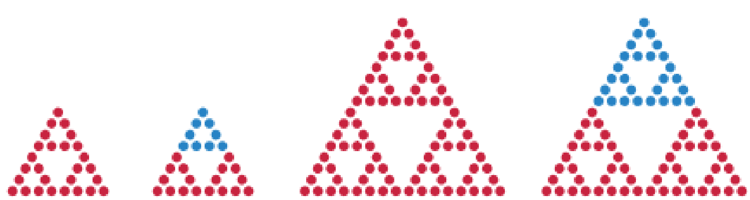
(c) Show that, for any prime number p, the sum of the interior numbers of the pth row is 2^p-2.

(d) Calculate 2^p-2 for $p=7$, and show that it is a multiple of 7.

(e) Use the results of parts (b) and (c) to show that, for any prime number p, 2^p-2 is a multiple of p. *Note:* This is a special case of Fermat's theorem, which states that, for any prime number p and any integer a, a^p-a is a multiple of p.

8. There are four odd numbers in the 6th row of Pascal's triangle (1, 15, 15, 1), and $4=2^2$ is a power of 2. For the 0th through 12th rows of Pascal's triangle, show that the number of odd numbers in each row is a power of 2.

In Fig. 1(a), each odd number in the first eight rows of Pascal's triangle has been replaced by a dot and each even number has been replaced by a space. In Fig. 1(b), the pattern for the first four rows is shown in blue. Notice that this pattern appears twice in the next four rows, with the two appearances separated by an inverted triangle. Fig. 1(c) shows the locations of the odd numbers in the first 16 rows of Pascal's triangle, and Fig. 1(d) shows that the pattern for the first eight rows appears twice in the next eight rows, separated by an inverted triangle. Figure 2 demonstrates that the first 32 rows of Pascal's triangle have the same property.



(a) (b) (c) (d)

Figure 1

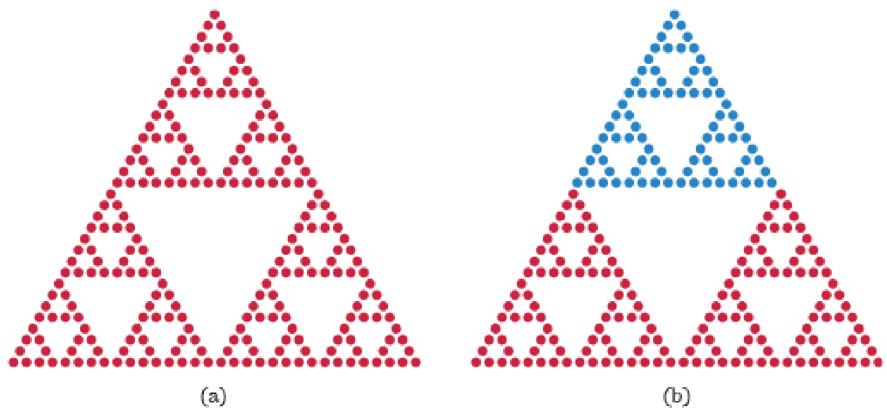


Figure 2

9. Assume that the property shown in Figs. 1 and 2 continues to hold for subsequent rows of Pascal's triangle. Use this result to explain why the number of odd numbers in each row of Pascal's triangle is a power of 2.

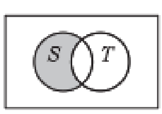
Selected Answers

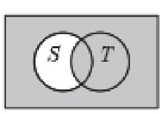
Exercises 1

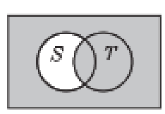
1. **(a)** $\{5, 6, 7\}$ **(b)** $\{1, 2, 3, 4, 5, 7\}$ **(c)** $\{1, 3\}$ **(d)** $\{5, 7\}$ **3.** **(a)** $\{a, b, c, e, i, o, u\}$ **(b)** $\{a\}$ **(c)** $\varnothing$ **(d)** $\{b, c\}$ **5.** $\varnothing, \{1\}, \{2\}, \{1, 2\}$ **7.** **(a)** {all freshman college students who like basketball} **(b)** {all college students who do not like basketball} **(c)** {all college students who are neither freshmen nor like basketball} **(d)** {all college students who are either freshmen or like basketball} **9.** **(a)** $S = \{1999, 2003, 2006, 2010, 2013\}$ **(b)** $T = \{1996, 1997, 1998, 1999, 2003, 2009, 2013\}$ **(c)** $S \cap T = \{1999, 2003, 2013\}$ **(d)** $S \cup T = \{1996, 1997, 1998, 1999, 2003, 2006, 2009, 2010, 2013\}$ **(e)** $S' \cap T = \{1996, 1997, 1998, 2009\}$ **(f)** $S \cap T' = \{2006, 2010\}$ **11.** Between 1996 and 2015, there were only two years in which the Standard and Poor's index increased by 2% or more during the first five days and did not increase by 16% or more for the entire year. **13.** **(a)** $\{e, f\}$ **(b)** $\{a, b, c, d, e, f\}$ **(c)** $\varnothing$ **(d)** $\{a, b\}$ **(e)** $\varnothing$ **(f)** $\{a, b, d, e, f\}$ **(g)** $\{a, b, c\}$ **(h)** $\{a, b\}$ **(i)** $\{d\}$ **15.** S **17.** U **19.** $\varnothing$ **21.** $L \cup T$ **23.** $L \cap P$ **25.** $P \cap L \cap T$ **27.** S' **29.** $S \cup A \cup D$ **31.** $(A \cap S)' \cap D$ **33.** {students at Mount College who are younger than 35} **35.** {people who are both students and teachers at Mount College} **37.** {people at Mount College who are students or are at most 35} **39.** {people at Mount College who are at least 35} **41.** V' **43.** $V \cap (C \cup S)'$ **45.** $(V \cup C)'$ **47.** **(a)** $\{B, C, D, E\}$ **(b)** $\{C, D, E, F\}$ **(c)** $\{A, D, E, F\}$ **(d)** $\{A, C, D, E, F\}$ **(e)** $\{A, F\}$ **(f)** $\{D, E\}$ **49.** 8 ways: no toppings; peppers; onions; mushrooms; peppers and onions; peppers and mushrooms; onions and mushrooms; all three toppings **51.** Possible answer: $\{2\}$ **53.** $S \subseteq T$ **55.** True **57.** True **59.** False **61.** True

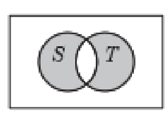
Exercises 2

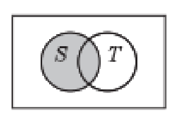
1. 6 **3.** 0 **5.** 8 **7.** $S \subseteq T$ **9.** 11 million **11.** 10 **13.** 452

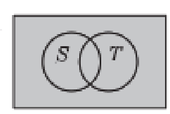
15. 

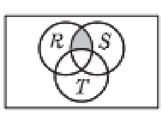
17. 

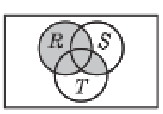
19. 

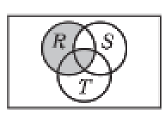
21. 

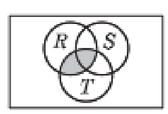
23. 

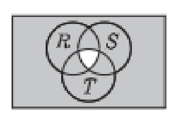
25. 

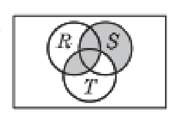
27. 

29. 

31. 

33. 

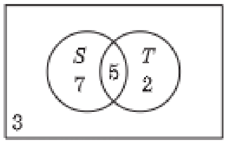
35. 

37. 

39. $S' \cup T'$ **41.** $S \cap T'$ **43.** U **45.** S' **47.** $R \cap T$ **49.** $R' \cap S \cap T$ **51.** $T \cup (R \cap S')$ **53.** $(R \cap S \cap T) \cup (R' \cap S' \cap T')$ **55.** Everyone who is not a citizen or is both over the age of 18 and employed **57.** Everyone over the age of 18 who is unemployed **59.** Noncitizens who are 5 years of age or older

Exercises 3

1. 11 **3.** 46 **5.** 11 **7.** 75 **9.** 30

11.


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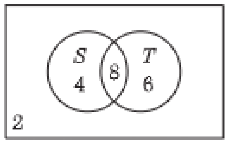

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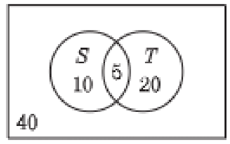

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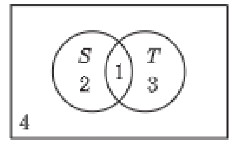

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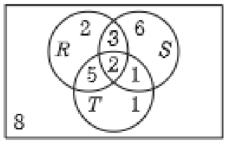

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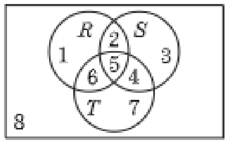
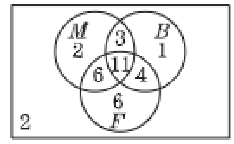


23. 25 **25.** 4; 2; 1 **27.** 55 **29.** 5 **31.** **(a)**


(b) 2 **(c)** 5 **33.** 28 **35.** 2 **37.** 51 **39.** 190 **41.** 180 **43.** 210 **45.** 100 **47.** 450 **49.** 3750 **51.** 30 **53.** 4 **55.** 49 **57.** 29 **59.** 26 **61.** 90 **63.** 6 **65.** 140 **67.** 30 **69.** 30 **71.** 3

Exercises 4

1. $4 \cdot 2 = 8$ **3.** $3 \cdot 2 = 6$ **5.** $44 \cdot 43 \cdot 42 = 79{,}464$ **7.** $20 \cdot 19 \cdot 18 = 6840$ **9.** 30, since $30 \cdot 29 = 870$
11. **(a)** $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$ **(b)** $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = 720$ **13.** $4 \cdot 3 \cdot 2 \cdot 1 = 24$ **15.** $2 \cdot 3 = 6$
17. $3 \cdot 12 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 360{,}000$ **19.** $10^9 - 1 = 999{,}999{,}999$ **21.** $8 \cdot 2 \cdot 10 = 160$ **23.** $9 \cdot 10 \cdot 10 \cdot 1 \cdot 1 = 900$ **25.** $26 \cdot 26 \cdot 1 \cdot 1 = 676$
27. $15 \cdot 15 = 225$ **29.** $3200 \cdot 2 \cdot 24 \cdot 52 = 7{,}987{,}200$ **31.** Since $26 \cdot 26 \cdot 26 = 17{,}576 < 20{,}000$, two students must have the same initials.
33. $7 \cdot 5 = 35$ **35.** $5 \cdot 4 = 20$ **37.** $2^6 = 64$ **39.** $2^5 = 32$ **41.** $4^{10} = 1{,}048{,}576$ **43.** $10^5 = 100{,}000$
45. $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$; one week **47.** $6 \cdot 7 \cdot 4 = 168$ days or 24 weeks **49.** $5 \cdot 11 \cdot (7 \cdot 2 + 1) \cdot 10 = 8250$ **51.** $2^4 = 16$
53. $2 \cdot 38 \cdot 38 = 2888$ **55.** **(a)** $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$ **(b)** $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 40{,}320$
(c) $1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 720$ **57.** $\frac{10 \cdot 9}{2} + 10 \cdot 10 = 145$ **59.** $4 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 972$ **61.** $7 \cdot 4 \cdot 2^6 = 1792$; $8 \cdot 5 \cdot 3^6 = 29{,}160$ **63.** $2^4 = 16$

Exercises 5

1. 12 **3.** 120 **5.** 120 **7.** 5 **9.** 7 **11.** n **13.** 1 **15.** $\frac{n(n-1)}{2}$ **17.** 720 **19.** 72 **21.** Permutation **23.** Combination
25. Neither **27.** $4! = 24$ **29.** $C(9, 7) = 36$ **31.** $C(8, 4) = 70$ **33.** $P(65, 5) = 991{,}186{,}560$ **35.** $C(10, 5) = 252$
37. $C(100, 3) = 161{,}700$; $C(7, 3) = 35$ **39.** $P(150, 3) = 3{,}307{,}800$ **41.** $C(52, 5) = 2{,}598{,}960$ **43.** $C(13, 5) = 1287$ **45.** $5! = 120$
47. **(a)** $C(10, 4) = 210$ **(b)** $C(10, 6) = 210$ **(c)** Selecting four sweaters to take is equivalent to selecting six sweaters to leave.
49. $C(8, 2) = 28$ **51.** $C(69, 5) \cdot 26 = 292{,}201{,}338$ **53.** **(b)** $\dfrac{C(59, 6)}{C(49, 6)} \approx 3.22$ **55.** Yes; Moe: $C(9, 2) = 36$; Joe: $C(7, 4) = 35$
57. $4! \cdot P(4, 3) \cdot P(5, 3) \cdot P(6, 3) \cdot P(7, 3) = 870{,}912{,}000$ **59.** $3! \cdot 3!^3 = 1296$ **61.** 10 **63.** $C(15, 3) + 15 \cdot 14 + 15 = 680$
65. $720 - 3! - 5! = 594$ **67.** **(a)** $C(45, 5) = 1{,}221{,}759$ **(b)** $C(100, 4) = 3{,}921{,}225$ **(c)** Lottery (a)
69. Yes; the number of ways to shuffle a deck of 52 cards is $52! \approx 8 \times 10^{67}$.

Exercises 6

1. **(a)** $2^8 = 256$ **(b)** $C(8, 4) = 70$ **3.** **(a)** $C(7, 5) + C(7, 6) + C(7, 7) = 29$ **(b)** $2^7 - 29 = 99$ **5.** $2 \cdot C(6, 3) = 40$; $2 \cdot C(5, 3) = 20$
7. $C(11, 5) \cdot C(6, 5) \cdot 1 = 2772$ **9.** $C(8, 5) = 56$ **11.** $C(7, 2) = 21$ **13.** $C(5, 2) \cdot C(4, 2) = 60$ **15.** $C(6, 2) = 15$
17. **(d)** $56 + 70 = 126$ **19.** $C(8, 3) = 56$ **21.** $C(10, 6) - C(7, 4) = 175$ **23.** **(a)** $C(12, 5) = 792$ **(b)** $C(7, 5) = 21$
(c) $C(7, 2) \cdot C(5, 3) = 210$ **(d)** $C(7, 4) \cdot 5 + 21 = 196$ **25.** **(a)** $C(10, 3) = 120$ **(b)** $C(8, 3) = 56$ **(c)** $120 - 56 = 64$
27. $C(4, 2) \cdot C(6, 2) = 90$ **29.** $C(4, 3) \cdot C(4, 2) = 24$ **31.** $13 \cdot C(4, 3) \cdot 12 \cdot C(4, 2) = 3744$ **33.** $C(7, 5) \cdot 5! \cdot 21 \cdot 20 = 1{,}058{,}400$
35. $C(10, 5) \cdot P(21, 5) = 615{,}353{,}760$ **37.** $6! \cdot 7 \cdot 3! = 30{,}240$ **39.** $C(9, 4) = 126$ **41.** $C(26, 22) \cdot C(10, 7) = 1{,}794{,}000$
43. $C(12, 6) = 924$ **45.** $C(100, 50)/2^{100} \approx 7.96\%$ **47.** $\dfrac{C(50, 10) \cdot C(50, 10)}{C(100, 20)} \approx 19.7\%$

Exercises 7

1. 153 **3.** 15 **5.** 8 **7.** 1 **9.** 1 **11.** n **13.** 1 **15.** $n!$ **17.** 64 **19.** 20 **21.** $x^{10} + 10x^9y + 45x^8y^2$ **23.** $105x^2y^{13} + 15xy^{14} + y^{15}$
25. $184{,}756x^{10}y^{10}$ **27.** 6 **29.** 330 **31.** $x^9 + 18x^8y + 144x^7y^2$ **33.** $673{,}596x^6y^6$ **35.** $-945x^4y^3$ **37.** $2^8 = 256$ **39.** $2^4 = 16$
41. $2^5 = 32$ **43.** $2^8 - 1 = 255$ **45.** $2 \cdot 3 \cdot 2^{13} = 49{,}152$ **47.** $2^7 - C(7, 6) - C(7, 7) = 120$ **49.** $2^8 - C(8, 1) - C(8, 0) = 247$
51. $[2^9 - C(9, 1) - C(9, 0)] \cdot [2^{10} - C(10, 1) - C(10, 0)] = 508{,}526$ **53.** No **57.** $2^{10} - 2 = 1022$

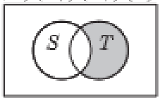
Exercises 8

1. 20 **3.** 180 **5.** 210 **7.** 34,650 **9.** 166,320 **11.** 1,401,400 **13.** 2,858,856 **15.** $\binom{20}{7, 5, 8} = 99{,}768{,}240$ **17.** $\binom{8}{2, 1, 4, 1} = 840$
19. $\binom{9}{3, 2, 4} = 1260$ **21.** $\dfrac{1}{5!} \cdot \dfrac{20!}{(4!)^5} = 2{,}546{,}168{,}625$ **23.** $\binom{30}{10, 2, 18} = 5{,}708{,}552{,}850$ **25.** $\binom{4}{1, 1, 2} = 12$ **27.** $\dfrac{1}{7!} \cdot \dfrac{14!}{(2!)^7} = 135{,}135$
29. $\dfrac{1}{2!} \cdot \dfrac{10!}{(5!)^2} = 126$ **31.** $n!$ **33.** $\binom{38}{10, 12, 10, 6} = 115{,}166{,}175{,}166{,}136{,}334{,}240$ **35.** Greater than; $\binom{52}{13, 13, 13, 13} \approx 5.4$ octillion

Chapter Review Exercises

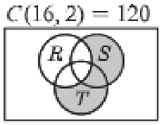
1. $\varnothing, \{a\}, \{b\}, \{a, b\}$

2.



3. $C(16, 2) = 120$ **4.** $2 \cdot 5! = 240$

5.



6. $x^{12} - 24x^{11}y + 264x^{10}y^2$

7. $C(8, 3) \cdot C(6, 2) = 840$ **8.** 15 **9.** $7 \cdot 5 = 35$ **10.** $\binom{12}{2, 4, 6} = 13{,}860$ **11.** 0 **12.** 136 **13.** 6 **14.** 49 **15.** 47 **16.** 11

17. 46 **18.** 58 **19.** 22 **20.** 23 **21.** $C(9, 4) = 126$ **22.** $2^{20} = 1{,}048{,}576$ **23.** 550 **24.** 480 **25.** $\binom{5}{1, 3, 1} = 20$

26. $9 \cdot 10^4 \cdot 1 \cdot 5 = 450{,}000$ **27.** $9 \cdot 9 \cdot 10^8 = 8{,}100{,}000{,}000$ **28.** $P(7, 3) = 210$ **29.** $5^8 = 390{,}625$; $5^8 - 4^8 = 325{,}089$

30. $C(12, 5) = 792$ **31.** $C(30, 14) = 145{,}422{,}675$ **32.** 27,500 **33.** $3^{10} = 59{,}049$ **34.** $C(60, 10) = 75{,}394{,}027{,}566$

35. $5^{10} = 9{,}765{,}625$ **36.** $21^6 = 85{,}766{,}121$ **37.** $C(10, 4) = 210$ **38.** $\frac{1}{3!} \cdot \frac{21!}{(7!)^3} = 66{,}512{,}160$ **39.** $14! = 87{,}178{,}291{,}200$

40. $\frac{1}{4!} \cdot \frac{20!}{(5!)^4} = 488{,}864{,}376$ **41.** $3 \cdot 5 \cdot 4 = 60$ **42.** $3 \cdot C(10, 2) = 135$ **43.** $C(n, 2) - n$ **44.** $8 \cdot 6 = 48$ **45.** $5! \cdot 4! \cdot 3! \cdot 2! \cdot 1! = 34{,}560$

46. $P(12, 5) = 95{,}040$ **47.** $4 \cdot C(13, 5) = 5148$ **48.** $C(4, 3) \cdot C(48, 2) = 4512$ **49.** $9 \cdot 10 \cdot 10 - 9 \cdot 9 \cdot 8 - 9 = 243$

50. $9 \cdot 9 \cdot 8 = 648$ **51.** $3 \cdot 3 = 9$ **52.** $2 \cdot 3! \cdot 3! = 72$ **53.** $24 \cdot 23 + 24 \cdot 23 \cdot 22 = 12{,}696$ **54.** $3 \cdot 4! \cdot 2 = 144$ **55.** $C(10, 2) = 45$

56. Second teacher: $\frac{1}{4!} \cdot \frac{24!}{(6!)^4} = 96{,}197{,}645{,}544$ vs $\frac{1}{6!} \cdot \frac{24!}{(4!)^6} = 4{,}509{,}264{,}634{,}875$ **57.** $\binom{10}{3, 4, 3} = 4200$

58. 5, since $5! = 120$ **59.** $7 \cdot 6 \cdot 5 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 5040$ **60.** **(a)** $C(12, 2) \cdot C(12, 3) = 14{,}520$ **(b)** $C(12, 5) \cdot 2^5 = 25{,}344$

61. $C(7, 2) + C(7, 1) + C(7, 0) = 29$ **62.** $2^6 = 64$ **63.** $2 \cdot 26 \cdot 26 + 2 \cdot 26 \cdot 26 \cdot 26 = 36{,}504$ **64.** 12

65. $\binom{25}{10, 9, 6} = 16{,}360{,}143{,}800$ **66.** $C(26, 3) \cdot C(10, 3) \cdot 6! = 224{,}640{,}000$