3.1 Exercises

1. (a) How is the number e defined?
   (b) Use a calculator to estimate the values of the limits

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{2.8^h - 1}{h}$$

   correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What fact allows you to do this?
   (b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.
   (c) Which of the two functions in part (b) grows more rapidly when x is large?

3–26 Differentiate the function.

3. $f(x) = 186.5$
4. $f(x) = \sqrt{30}$
5. $f(t) = 2 - \frac{2}{3}t$
6. $F(x) = \frac{3}{4}x^8$
7. $f(x) = x^3 - 4x + 6$
8. $f(t) = \frac{1}{2}t^6 - 3t^4 + t$
9. $f(t) = \frac{1}{4}(t^4 + 8)$
10. $h(x) = (x - 2)(2x + 3)$
11. $A(s) = -\dfrac{12}{s^5}$
12. $B(y) = cy^{-6}$
13. $g(t) = 2t^{-3/4}$
14. $h(t) = \sqrt[4]{t} - 4e^t$
15. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}}$
16. $y = \sqrt{x}\,(x - 1)$
17. $F(x) = \left(\frac{1}{2}x\right)^5$
18. $f(x) = \dfrac{x^2 - 3x + 1}{x^2}$
19. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$
20. $g(u) = \sqrt{2}\,u + \sqrt{3u}$
21. $y = 4\pi^2$
22. $y = ae^v + \dfrac{b}{v} + \dfrac{c}{v^2}$
23. $u = \sqrt[5]{t} + 4\sqrt{t^5}$
24. $v = \left(\sqrt{x} + \dfrac{1}{\sqrt[3]{x}}\right)^2$
25. $z = \dfrac{A}{y^{10}} + Be^y$
26. $y = e^{x+1} + 1$

27–28 Find an equation of the tangent line to the curve at the given point.

27. $y = \sqrt[4]{x}$, $(1, 1)$
28. $y = x^4 + 2x^2 - x$, $(1, 2)$

29–30 Find equations of the tangent line and normal line to the curve at the given point.

29. $y = x^4 + 2e^x$, $(0, 2)$
30. $y = (1 + 2x)^2$, $(1, 9)$

31–32 Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

31. $y = 3x^2 - x^3$, $(1, 2)$
32. $y = x - \sqrt{x}$, $(1, 0)$

33–36 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

33. $f(x) = e^x - 5x$
34. $f(x) = 3x^5 - 20x^3 + 50x$
35. $f(x) = 3x^{15} - 5x^3 + 3$
36. $f(x) = x + \dfrac{1}{x}$

37–38 Estimate the value of $f'(a)$ by zooming in on the graph of f. Then differentiate f to find the exact value of $f'(a)$ and compare with your estimate.

37. $f(x) = 3x^2 - x^3$, $a = 1$
38. $f(x) = 1/\sqrt{x}$, $a = 4$

39. (a) Use a graphing calculator or computer to graph the function $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$ in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.
(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 1 in Section 2.7.)
(c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (b).

40. (a) Use a graphing calculator or computer to graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.
(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g'. (See Example 1 in Section 2.7.)
(c) Calculate $g'(x)$ and use this expression, with a graphing device, to graph g'. Compare with your sketch in part (b).

41–42 Find the first and second derivatives of the function.

41. $f(x) = 10x^{10} + 5x^5 - x$

42. $G(r) = \sqrt{r} + \sqrt[3]{r}$

43–44 Find the first and second derivatives of the function. Check to see that your answers are reasonable by comparing the graphs of f, f', and f''.

43. $f(x) = 2x - 5x^{3/4}$

44. $f(x) = e^x - x^3$

45. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find
(a) the velocity and acceleration as functions of t,
(b) the acceleration after 2 s, and
(c) the acceleration when the velocity is 0.

46. The equation of motion of a particle is $s = t^4 - 2t^3 + t^2 - t$, where s is in meters and t is in seconds.
(a) Find the velocity and acceleration as functions of t.
(b) Find the acceleration after 1 s.
(c) Graph the position, velocity, and acceleration functions on the same screen.

47. On what interval is the function $f(x) = 5x - e^x$ increasing?

48. On what interval is the function $f(x) = x^3 - 4x^2 + 5x$ concave upward?

49. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

50. For what values of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

51. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

52. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

53. Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and parallel to the line $12x - y = 1$.

54. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

55. Find an equation of the normal line to the parabola $y = x^2 - 5x + 4$ that is parallel to the line $x - 3y = 5$.

56. Where does the normal line to the parabola $y = x - x^2$ at the point $(1, 0)$ intersect the parabola a second time? Illustrate with a sketch.

57. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

58. (a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.
(b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.

59. Use the definition of a derivative to show that if $f(x) = 1/x$, then $f'(x) = -1/x^2$. (This proves the Power Rule for the case $n = -1$.)

60. Find the nth derivative of each function by calculating the first few derivatives and observing the pattern that occurs.
(a) $f(x) = x^n$
(b) $f(x) = 1/x$

61. Find a second-degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

62. The equation $y'' + y' - 2y = x^2$ is called a **differential equation** because it involves an unknown function y and its derivatives y' and y''. Find constants A, B, and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation. (Differential equations will be studied in detail in Chapter 7.)

63. (a) In Section 2.8 we defined an antiderivative of f to be a function F such that $F' = f$. Try to guess a formula for an antiderivative of $f(x) = x^2$. Then check your answer by differentiating it. How many antiderivatives does f have?
(b) Find antiderivatives for $f(x) = x^3$ and $f(x) = x^4$.
(c) Find an antiderivative for $f(x) = x^n$, where $n \neq -1$. Check by differentiation.

64. Use the result of Exercise 63(c) to find an antiderivative of each function.
(a) $f(x) = \sqrt{x}$
(b) $f(x) = e^x + 8x^3$

65. Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$.

66. Suppose the curve $y = x^4 + ax^3 + bx^2 + cx + d$ has a tangent line when $x = 0$ with equation $y = 2x + 1$ and a tangent line when $x = 1$ with equation $y = 2 - 3x$. Find the values of a, b, c, and d.

67. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

68. Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$.

69. For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

70. A tangent line is drawn to the hyperbola $xy = c$ at a point P.
(a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P.
(b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.

71. Evaluate $\lim_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$.

72. Draw a diagram showing two perpendicular lines that intersect on the y-axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

73. If $c > \frac{1}{2}$, how many lines through the point $(0, c)$ are normal lines to the parabola $y = x^2$? What if $c \leqslant \frac{1}{2}$?

74. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find its equation. If not, why not?

3.2 Exercises

1. Find the derivative of $f(x) = (1 + 2x^2)(x - x^2)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

2. Find the derivative of the function

$$F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

3–24 Differentiate.

3. $f(x) = (x^3 + 2x)e^x$

4. $g(x) = \sqrt{x}\, e^x$

5. $y = \dfrac{e^x}{x^2}$

6. $y = \dfrac{e^x}{1 + x}$

7. $g(x) = \dfrac{3x - 1}{2x + 1}$

8. $f(t) = \dfrac{2t}{4 + t^2}$

9. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3)$

10. $R(t) = (t + e^t)(3 - \sqrt{t})$

11. $y = \dfrac{x^3}{1 - x^2}$

12. $y = \dfrac{x + 1}{x^3 + x - 2}$

13. $y = \dfrac{t^2 + 2}{t^4 - 3t^2 + 1}$

14. $y = \dfrac{t}{(t - 1)^2}$

Graphing calculator or computer with graphing software required

1. Homework Hints available in TEC

15. $y = (r^2 - 2r)e^r$

16. $y = \dfrac{1}{s + ke^s}$

17. $y = \dfrac{v^3 - 2v\sqrt{v}}{v}$

18. $z = w^{3/2}(w + ce^w)$

19. $f(t) = \dfrac{2t}{2 + \sqrt{t}}$

20. $g(t) = \dfrac{t - \sqrt{t}}{t^{1/3}}$

21. $f(x) = \dfrac{A}{B + Ce^x}$

22. $f(x) = \dfrac{1 - xe^x}{x + e^x}$

23. $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$

24. $f(x) = \dfrac{ax + b}{cx + d}$

25–28 Find $f'(x)$ and $f''(x)$.

25. $f(x) = x^4e^x$

26. $f(x) = x^{5/2}e^x$

27. $f(x) = \dfrac{x^2}{1 + 2x}$

28. $f(x) = \dfrac{x}{x^2 - 1}$

29–30 Find an equation of the tangent line to the given curve at the specified point.

29. $y = \dfrac{2x}{x + 1}$, $(1, 1)$

30. $y = \dfrac{e^x}{x}$, $(1, e)$

31–32 Find equations of the tangent line and normal line to the given curve at the specified point.

31. $y = 2xe^x$, $(0, 0)$

32. $y = \dfrac{\sqrt{x}}{x + 1}$, $(4, 0.4)$

33. (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $\left(-1, \frac{1}{2}\right)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

34. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point $(3, 0.3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

35. (a) If $f(x) = (x^3 - x)e^x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

36. (a) If $f(x) = e^x/(2x^2 + x + 1)$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

37. (a) If $f(x) = (x^2 - 1)/(x^2 + 1)$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

38. (a) If $f(x) = (x^2 - 1)e^x$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

39. If $f(x) = x^2/(1 + x)$, find $f''(1)$.

40. If $g(x) = x/e^x$, find $g^{(n)}(x)$.

41. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.
(a) $(fg)'(5)$
(b) $(f/g)'(5)$
(c) $(g/f)'(5)$

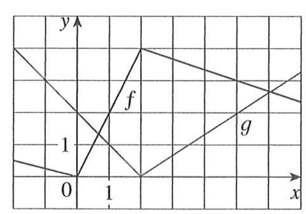
42. Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$.
(a) $h(x) = 5f(x) - 4g(x)$
(b) $h(x) = f(x)g(x)$
(c) $h(x) = \dfrac{f(x)}{g(x)}$
(d) $h(x) = \dfrac{g(x)}{1 + f(x)}$

43. If $f(x) = e^x g(x)$, where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

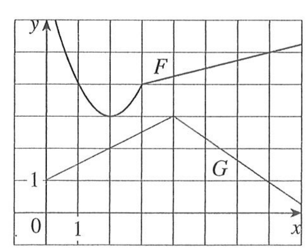
44. If $h(2) = 4$ and $h'(2) = -3$, find

$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\bigg|_{x=2}$$

45. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.
(a) Find $u'(1)$.
(b) Find $v'(5)$.



46. Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where F and G are the functions whose graphs are shown.
(a) Find $P'(2)$.
(b) Find $Q'(7)$.



47. If g is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = xg(x)$ (b) $y = \dfrac{x}{g(x)}$ (c) $y = \dfrac{g(x)}{x}$

48. If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2 f(x)$ (b) $y = \dfrac{f(x)}{x^2}$

(c) $y = \dfrac{x^2}{f(x)}$ (d) $y = \dfrac{1 + xf(x)}{\sqrt{x}}$

49. In this exercise we estimate the rate at which the total personal income is rising in the Richmond-Petersburg, Virginia, metropolitan area. In 1999, the population of this area was 961,400, and the population was increasing at roughly 9200 people per year. The average annual income was \$30,593 per capita, and this average was increasing at about \$1400 per year (a little above the national average of about \$1225 yearly). Use the Product Rule and these figures to estimate the rate at which total personal income was rising in the Richmond-Petersburg area in 1999. Explain the meaning of each term in the Product Rule.

50. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.
(a) What does it mean to say that $f(20) = 10{,}000$ and $f'(20) = -350$?
(b) Assuming the values in part (a), find $R'(20)$ and interpret your answer.

51. On what interval is the function $f(x) = x^3e^x$ increasing?

52. On what interval is the function $f(x) = x^2e^x$ concave downward?

53. How many tangent lines to the curve $y = x/(x + 1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

54. Find equations of the tangent lines to the curve

$$y = \frac{x - 1}{x + 1}$$

that are parallel to the line $x - 2y = 2$.

55. Find $R'(0)$, where

$$R(x) = \frac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9}$$

Hint: Instead of finding $R'(x)$ first, let $f(x)$ be the numerator and $g(x)$ the denominator of $R(x)$ and compute $R'(0)$ from $f(0)$, $f'(0)$, $g(0)$, and $g'(0)$.

56. Use the method of Exercise 55 to compute $Q'(0)$, where

$$Q(x) = \frac{1 + x + x^2 + xe^x}{1 - x + x^2 - xe^x}$$

57. (a) Use the Product Rule twice to prove that if f, g, and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.
(b) Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(c) Use part (b) to differentiate $y = e^{3x}$.

58. (a) If $F(x) = f(x)g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.
(b) Find similar formulas for F''' and $F^{(4)}$.
(c) Guess a formula for $F^{(n)}$.

59. Find expressions for the first five derivatives of $f(x) = x^2e^x$. Do you see a pattern in these expressions? Guess a formula for $f^{(n)}(x)$ and prove it using mathematical induction.

60. (a) If g is differentiable, the **Reciprocal Rule** says that

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{[g(x)]^2}$$

Use the Quotient Rule to prove the Reciprocal Rule.
(b) Use the Reciprocal Rule to differentiate the function in Exercise 16.
(c) Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is,

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

for all positive integers n.

3.4 Exercises

1–6 Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

1. $y = \sqrt[3]{1 + 4x}$

2. $y = (2x^3 + 5)^4$

3. $y = \tan \pi x$

4. $y = \sin(\cot x)$

5. $y = e^{\sqrt{x}}$

6. $y = \sqrt{2 - e^x}$

7–36 Find the derivative of the function.

7. $F(x) = (x^4 + 3x^2 - 2)^5$

8. $F(x) = (4x - x^2)^{100}$

9. $F(x) = \sqrt{1 - 2x}$

10. $f(x) = (1 + x^4)^{2/3}$

11. $f(z) = \dfrac{1}{z^2 + 1}$

12. $f(t) = \sqrt[3]{1 + \tan t}$

13. $y = \cos(a^3 + x^3)$

14. $y = a^3 + \cos^3 x$

15. $h(t) = t^3 - 3^t$

16. $y = 3\cot(n\theta)$

17. $y = xe^{-kx}$

18. $y = e^{-2t}\cos 4t$

19. $y = (2x - 5)^4(8x^2 - 5)^{-3}$

20. $h(t) = (t^4 - 1)^3(t^3 + 1)^4$

21. $y = e^{x\cos x}$

22. $y = 10^{1-x^2}$

23. $y = \left(\dfrac{x^2 + 1}{x^2 - 1}\right)^3$

24. $G(y) = \left(\dfrac{y^2}{y + 1}\right)^5$

25. $y = \sec^2 x + \tan^2 x$

26. $y = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$

27. $y = \dfrac{r}{\sqrt{r^2 + 1}}$

28. $y = e^{k\tan\sqrt{x}}$

29. $y = \sin(\tan 2x)$

30. $f(t) = \sqrt{\dfrac{t}{t^2 + 4}}$

31. $y = 2^{\sin \pi x}$

32. $y = \sin(\sin(\sin x))$

33. $y = \cot^2(\sin\theta)$

34. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

35. $y = \cos\sqrt{\sin(\tan \pi x)}$

36. $y = 2^{3^{x^2}}$

37–40 Find y' and y''.

37. $y = \cos(x^2)$

38. $y = \cos^2 x$

39. $y = e^{\alpha x}\sin\beta x$

40. $y = e^{e^x}$

41–44 Find an equation of the tangent line to the curve at the given point.

41. $y = (1 + 2x)^{10}$, $(0, 1)$

42. $y = \sqrt{1 + x^3}$, $(2, 3)$

43. $y = \sin(\sin x)$, $(\pi, 0)$

44. $y = \sin x + \sin^2 x$, $(0, 0)$

45. (a) Find an equation of the tangent line to the curve $y = 2/(1 + e^{-x})$ at the point $(0, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

46. (a) The curve $y = |x|/\sqrt{2 - x^2}$ is called a *bullet-nose curve*. Find an equation of the tangent line to this curve at the point $(1, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

47. (a) If $f(x) = x\sqrt{2 - x^2}$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

48. The function $f(x) = \sin(x + \sin 2x)$, $0 \leqslant x \leqslant \pi$, arises in applications to frequency modulation (FM) synthesis.
(a) Use a graph of f produced by a graphing device to make a rough sketch of the graph of f'.
(b) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (a).

49. Find all points on the graph of the function $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

50. Find the x-coordinates of all points on the curve $y = \sin 2x - 2\sin x$ at which the tangent line is horizontal.

51. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

52. If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

Graphing calculator or computer with graphing software required   CAS Computer algebra system required   **1.** Homework Hints available in TEC

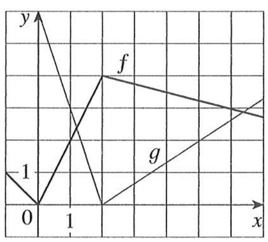
53. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

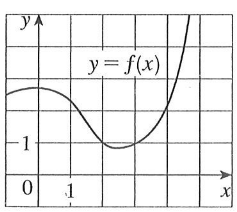
(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

54. Let f and g be the functions in Exercise 53.
(a) If $F(x) = f(f(x))$, find $F'(2)$.
(b) If $G(x) = g(g(x))$, find $G'(3)$.

55. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.
(a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$



56. If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.
(a) $h'(2)$ (b) $g'(2)$



57. Use the table to estimate the value of $h'(0.5)$, where $h(x) = f(g(x))$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	12.6	14.8	18.4	23.0	25.9	27.5	29.1
$g(x)$	0.58	0.40	0.37	0.26	0.17	0.10	0.05

58. If $g(x) = f(f(x))$, use the table to estimate the value of $g'(1)$.

x	0.0	0.5	1.0	1.5	2.0	2.5
$f(x)$	1.7	1.8	2.0	2.4	3.1	4.4

59. Suppose f is differentiable on $\mathbb{R}$. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

60. Suppose f is differentiable on $\mathbb{R}$ and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

61. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

62. If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

63. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

64. If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

65. Show that the function $y = e^{2x}(A \cos 3x + B \sin 3x)$ satisfies the differential equation $y'' - 4y' + 13y = 0$.

66. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' - 4y' + y = 0$?

67. Find the 50th derivative of $y = \cos 2x$.

68. Find the 1000th derivative of $f(x) = xe^{-x}$.

69. The displacement of a particle on a vibrating string is given by the equation

$$s(t) = 10 + \tfrac{1}{4}\sin(10\pi t)$$

where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

70. If the equation of motion of a particle is given by $s = A\cos(\omega t + \delta)$, the particle is said to undergo *simple harmonic motion.*
(a) Find the velocity of the particle at time t.
(b) When is the velocity 0?

71. A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by ± 0.35. In view of these data, the brightness of Delta Cephei at time t, where t is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right)$$

(a) Find the rate of change of the brightness after t days.
(b) Find, correct to two decimal places, the rate of increase after one day.

72. In Example 4 in Section 1.3 we arrived at a model for the length of daylight (in hours) in Philadelphia on the tth day of the year:

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Use this model to compare how the number of hours of daylight is increasing in Philadelphia on March 21 and May 21.

73. The motion of a spring that is subject to a frictional force or a damping force (such as a shock absorber in a car) is often modeled by the product of an exponential function and a sine or cosine function. Suppose the equation of motion of a point on such a spring is

$$s(t) = 2e^{-1.5t} \sin 2\pi t$$

where s is measured in centimeters and t in seconds. Find the velocity after t seconds and graph both the position and velocity functions for $0 \leqslant t \leqslant 2$.

74. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 7.5 we will see that this is a reasonable equation for $p(t)$.]

(a) Find $\lim_{t \to \infty} p(t)$.

(b) Find the rate of spread of the rumor.

(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

75. A particle moves along a straight line with displacement $s(t)$, velocity $v(t)$, and acceleration $a(t)$. Show that

$$a(t) = v(t) \frac{dv}{ds}$$

Explain the difference between the meanings of the derivatives dv/dt and dv/ds.

76. Air is being pumped into a spherical weather balloon. At any time t, the volume of the balloon is $V(t)$ and its radius is $r(t)$.

(a) What do the derivatives dV/dr and dV/dt represent?

(b) Express dV/dt in terms of dr/dt.

77. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a) Use a graphing calculator or computer to find an exponential model for the charge.

(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 2 in Section 2.1.

78. The table gives the US population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?

(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.

(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).

(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

79–81 Find an equation of the tangent line to the curve at the point corresponding to the given value of the parameter.

79. $x = t^4 + 1, \quad y = t^3 + t; \quad t = -1$

80. $x = \cos\theta + \sin 2\theta, \quad y = \sin\theta + \cos 2\theta; \quad \theta = 0$

81. $x = e^{\sqrt{t}}, \quad y = t - \ln t^2; \quad t = 1$

82–83 Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

82. $x = 2t^3 + 3t^2 - 12t, \quad y = 2t^3 + 3t^2 + 1$

83. $x = 10 - t^2, \quad y = t^3 - 12t$

84. Show that the curve with parametric equations $x = \sin t$, $y = \sin(t + \sin t)$ has two tangent lines at the origin and find their equations. Illustrate by graphing the curve and its tangents.

85. A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point (3, 0) and find their equations.

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Illustrate parts (a) and (b) by graphing C and the tangent lines.

86. The cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ was discussed in Example 7 in Section 1.7.

(a) Find an equation of the tangent to the cycloid at the point where $\theta = \pi/3$.

(b) At what points is the tangent horizontal? Where is it vertical?

(c) Graph the cycloid and its tangent lines for the case $r = 1$.

CAS **87.** Computer algebra systems have commands that differentiate functions, but the form of the answer may not be convenient and so further commands may be necessary to simplify the answer.

(a) Use a CAS to find the derivative in Example 5 and compare with the answer in that example. Then use the simplify command and compare again.

(b) Use a CAS to find the derivative in Example 6. What happens if you use the simplify command? What happens if you use the factor command? Which form of the answer would be best for locating horizontal tangents?

CAS **88.** (a) Use a CAS to differentiate the function

$$f(x) = \sqrt{\frac{x^4 - x + 1}{x^4 + x + 1}}$$

and to simplify the result.

(b) Where does the graph of f have horizontal tangents?

(c) Graph f and f' on the same screen. Are the graphs consistent with your answer to part (b)?

89. (a) If n is a positive integer, prove that

$$\frac{d}{dx}(\sin^n x \cos nx) = n \sin^{n-1} x \cos(n + 1)x$$

(b) Find a formula for the derivative of $y = \cos^n x \cos nx$ that is similar to the one in part (a).

90. Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point (4, 3).

91. Use the Chain Rule to show that if θ is measured in degrees, then

$$\frac{d}{d\theta}(\sin \theta) = \frac{\pi}{180} \cos \theta$$

(This gives one reason for the convention that radian measu is always used when dealing with trigonometric functions ir calculus: The differentiation formulas would not be as simp if we used degree measure.)

92. (a) Write $|x| = \sqrt{x^2}$ and use the Chain Rule to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

(b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f'. Where is f not differentiable?

(c) If $g(x) = \sin |x|$, find $g'(x)$ and sketch the graphs of g and g'. Where is g not differentiable?

93. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

94. Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hours for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely?

3.7 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–20 Differentiate the function.

2. $f(x) = x \ln x - x$

3. $f(x) = \sin(\ln x)$

4. $f(x) = \ln(\sin^2 x)$

5. $f(x) = \log_2(1 - 3x)$

6. $f(x) = \log_5(xe^x)$

7. $f(x) = \sqrt[5]{\ln x}$

8. $f(x) = \ln \sqrt[5]{x}$

9. $f(x) = \sin x \ln(5x)$

10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

11. $F(t) = \ln \dfrac{(2t + 1)^3}{(3t - 1)^4}$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $g(x) = \ln\left(x\sqrt{x^2 - 1}\right)$

14. $F(y) = y \ln(1 + e^y)$

15. $y = \ln |2 - x - 5x^2|$

16. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

17. $y = \ln(e^{-x} + xe^{-x})$

18. $y = [\ln(1 + e^x)]^2$

19. $y = 2x \log_{10} \sqrt{x}$

20. $y = \log_2(e^{-x} \cos \pi x)$

21–22 Find y' and y''.

21. $y = x^2 \ln(2x)$

22. $y = \dfrac{\ln x}{x^2}$

23–24 Differentiate f and find the domain of f.

23. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

24. $f(x) = \ln \ln \ln x$

25–27 Find an equation of the tangent line to the curve at the given point.

25. $y = \ln(x^2 - 3x + 1)$, $(3, 0)$

26. $y = \ln(x^3 - 7)$, $(2, 0)$

27. $y = \ln\left(xe^{x^2}\right)$, $(1, 1)$

28. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

29. (a) On what interval is $f(x) = x \ln x$ decreasing?
(b) On what interval is f concave upward?

30. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

31. Let $f(x) = cx + \ln(\cos x)$. For what value of c is $f'(\pi/4) = 6$?

32. Let $f(x) = \log_a(3x^2 - 2)$. For what value of a is $f'(1) = 3$?

33–42 Use logarithmic differentiation to find the derivative of the function.

33. $y = (2x + 1)^5(x^4 - 3)^6$

34. $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$

35. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

36. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

37. $y = x^x$

38. $y = x^{\cos x}$

39. $y = (\cos x)^x$

40. $y = \sqrt{x}^{\,x}$

41. $y = (\tan x)^{1/x}$

42. $y = (\sin x)^{\ln x}$

43. Find y' if $y = \ln(x^2 + y^2)$.

44. Find y' if $x^y = y^x$.

45. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

46. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

47. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

48. Show that $\displaystyle\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

Graphing calculator or computer with graphing software required (exercises 28 and 30)

1. Homework Hints available in TEC

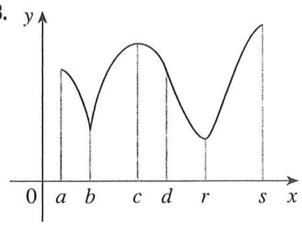
4.2 Exercises

1. Explain the difference between an absolute minimum and a local minimum.

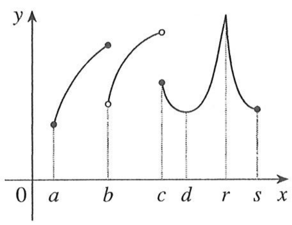
2. Suppose f is a continuous function defined on a closed interval $[a, b]$.
(a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?
(b) What steps would you take to find those maximum and minimum values?

3–4 For each of the numbers a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.

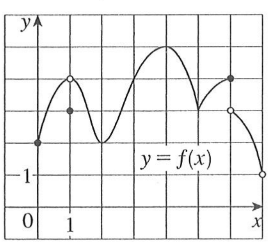


4.

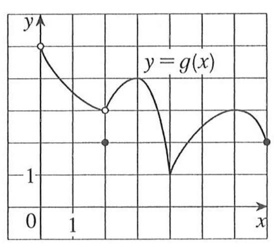


5–6 Use the graph to state the absolute and local maximum and minimum values of the function.

5.



6.



7–10 Sketch the graph of a function f that is continuous on $[1, 5]$ and has the given properties.

7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4

8. Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4

9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

10. f has no local maximum or minimum, but 2 and 4 are critical numbers

11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
(b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
(c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

12. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no local maximum.
(b) Sketch the graph of a function on $[-1, 2]$ that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.
(b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–22 Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Sections 1.2 and 1.3.)

15. $f(x) = \frac{1}{2}(3x - 1), \quad x \leqslant 3$

16. $f(x) = 2 - \frac{1}{3}x, \quad x \geqslant -2$

17. $f(x) = x^2, \quad 0 < x < 2$

18. $f(x) = e^x$

19. $f(x) = \ln x, \quad 0 < x \leqslant 2$

20. $f(t) = \cos t, \quad -3\pi/2 \leqslant t \leqslant 3\pi/2$

21. $f(x) = 1 - \sqrt{x}$

22. $f(x) = \begin{cases} 4 - x^2 & \text{if } -2 \leqslant x < 0 \\ 2x - 1 & \text{if } 0 \leqslant x \leqslant 2 \end{cases}$

23–38 Find the critical numbers of the function.

23. $f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2$

24. $f(x) = x^3 + 6x^2 - 15x$

25. $f(x) = x^3 + 3x^2 - 24x$

26. $f(x) = x^3 + x^2 + x$

27. $s(t) = 3t^4 + 4t^3 - 6t^2$

28. $g(t) = |3t - 4|$

29. $g(y) = \dfrac{y - 1}{y^2 - y + 1}$

30. $h(p) = \dfrac{p - 1}{p^2 + 4}$

31. $h(t) = t^{3/4} - 2t^{1/4}$

32. $g(x) = x^{1/3} - x^{-2/3}$

33. $F(x) = x^{4/5}(x - 4)^2$

34. $g(\theta) = 4\theta - \tan\theta$

35. $f(\theta) = 2\cos\theta + \sin^2\theta$

36. $h(t) = 3t - \arcsin t$

37. $f(x) = x^2 e^{-3x}$

38. $f(x) = x^{-2}\ln x$

Graphing calculator or computer with graphing software required

1. Homework Hints available in TEC

39–40 A formula for the *derivative* of a function f is given. How many critical numbers does f have?

39. $f'(x) = 5e^{-0.1|x|}\sin x - 1$

40. $f'(x) = \dfrac{100\cos^2 x}{10 + x^2} - 1$

41–54 Find the absolute maximum and absolute minimum values of f on the given interval.

41. $f(x) = 12 + 4x - x^2$, $[0, 5]$

42. $f(x) = 5 + 54x - 2x^3$, $[0, 4]$

43. $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

44. $f(x) = x^3 - 6x^2 + 9x + 2$, $[-1, 4]$

45. $f(x) = x^4 - 2x^2 + 3$, $[-2, 3]$

46. $f(x) = (x^2 - 1)^3$, $[-1, 2]$

47. $f(t) = t\sqrt{4 - t^2}$, $[-1, 2]$

48. $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$, $[-4, 4]$

49. $f(x) = xe^{-x^2/8}$, $[-1, 4]$

50. $f(x) = x - \ln x$, $[\frac{1}{2}, 2]$

51. $f(x) = \ln(x^2 + x + 1)$, $[-1, 1]$

52. $f(x) = x - 2\tan^{-1}x$, $[0, 4]$

53. $f(t) = 2\cos t + \sin 2t$, $[0, \pi/2]$

54. $f(t) = t + \cot(t/2)$, $[\pi/4, 7\pi/4]$

55. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1 - x)^b$, $0 \leqslant x \leqslant 1$.

56. Use a graph to estimate the critical numbers of $f(x) = |x^3 - 3x^2 + 2|$ correct to one decimal place.

57–60

(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.

(b) Use calculus to find the exact maximum and minimum values.

57. $f(x) = x^5 - x^3 + 2$, $-1 \leqslant x \leqslant 1$

58. $f(x) = e^{x^3 - x}$, $-1 \leqslant x \leqslant 0$

59. $f(x) = x\sqrt{x - x^2}$

60. $f(x) = x - 2\cos x$, $-2 \leqslant x \leqslant 0$

61. Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given approximately by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which water has its maximum density.

62. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu\sin\theta + \cos\theta}$$

where μ is a positive constant called the *coefficient of friction* and where $0 \leqslant \theta \leqslant \pi/2$. Show that F is minimized when $\tan\theta = \mu$.

63. A model for the US average price of a pound of white sugar from 1993 to 2003 is given by the function

$$S(t) = -0.00003237t^5 + 0.0009037t^4 - 0.008956t^3 + 0.03629t^2 - 0.04458t + 0.4074$$

where t is measured in years since August of 1993. Estimate the times when sugar was cheapest and most expensive during the period 1993–2003.

64. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

(a) Use a graphing calculator or computer to find the cubic polynomial that best models the velocity of the shuttle for the time interval $t \in [0, 125]$. Then graph this polynomial.

(b) Find a model for the acceleration of the shuttle and use it to estimate the maximum and minimum values of the acceleration during the first 125 seconds.

65. When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity v of the airstream is related to the radius r of the trachea by

V EXAMPLE 6 Maximizing revenue A store has been selling 200 DVD burners a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

SOLUTION If x is the number of DVD burners sold per week, then the weekly increase in sales is $x - 200$. For each increase of 20 units sold, the price is decreased by \$10. So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand function is

$$p(x) = 350 - \tfrac{10}{20}(x - 200) = 450 - \tfrac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \tfrac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$. This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward). The corresponding price is

$$p(450) = 450 - \tfrac{1}{2}(450) = 225$$

and the rebate is $350 - 225 = 125$. Therefore, to maximize revenue, the store should offer a rebate of \$125.

4.6 Exercises

1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.
   (a) Make a table of values, like the following one, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.

First number	Second number	Product
1	22	22
2	21	42
3	20	60
⋮	⋮	⋮

   (b) Use calculus to solve the problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

7. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

   where k is a positive constant. What nitrogen level gives the best yield?

8. The rate (in mg carbon/m^3/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

   where I is the light intensity (measured in thousands of foot-candles). For what light intensity is P a maximum?

9. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
   (a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
   (c) Write an expression for the total area.
   (d) Use the given information to write an equation that relates the variables.
   (e) Use part (d) to write the total area as a function of one variable.
   (f) Finish solving the problem and compare the answer with your estimate in part (a).

10. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
   (a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
   (c) Write an expression for the volume.
   (d) Use the given information to write an equation that relates the variables.
   (e) Use part (d) to write the volume as a function of one variable.
   (f) Finish solving the problem and compare the answer with your estimate in part (a).

11. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

12. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

13. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
   (b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

14. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

15. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

16. Find, correct to two decimal places, the coordinates of the point on the curve $y = \tan x$ that is closest to the point $(1, 1)$.

17. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

18. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

19. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

20. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

21. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

22. A cylindrical can without a top is made to contain V cm^3 of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

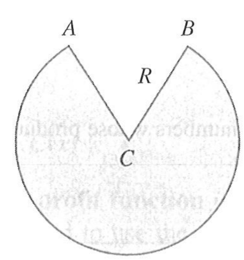
23. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 58 on page 24.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

24. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

25. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

26. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

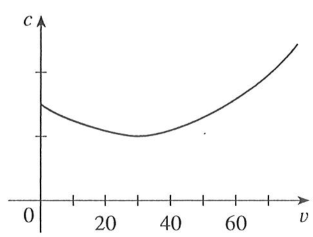
27. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



28. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

29. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

30. The graph shows the fuel consumption c of a car (measured in gallons per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 30$ mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons *per mile*. Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.



31. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

32. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u $(u < v)$, then the time required to swim a distance L is $L/(v - u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

where a is the proportionality constant.

(a) Determine the value of v that minimizes E.

(b) Sketch the graph of E.

Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

33. In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end as in the figure. It is believed that bees form their cells in such a way as to minimize the surface area for a given volume, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent. Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S = 6sh - \tfrac{3}{2}s^2 \cot\theta + \left(3s^2\sqrt{3}/2\right)\csc\theta$$

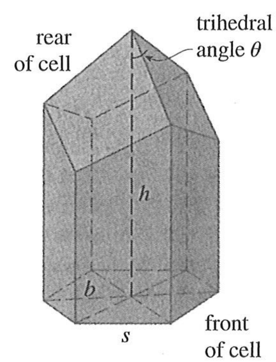
where s, the length of the sides of the hexagon, and h, the height, are constants.

(a) Calculate $dS/d\theta$.

(b) What angle should the bees prefer?

(c) Determine the minimum surface area of the cell (in terms of s and h).

Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than 2°.



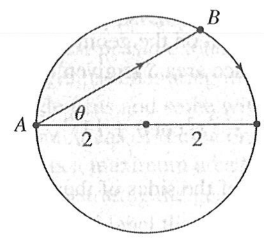
34. A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

35. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000/km over land to a point P on the north bank and \$800,000/km under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

36. Suppose the refinery in Exercise 35 is located 1 km north of the river. Where should P be located?

37. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, where should an object be placed on the line between the sources so as to receive the least illumination?

38. A woman at a point A on the shore of a circular lake with radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time (see the figure). She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. How should she proceed?



39. Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.

40. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

41. What is the shortest possible length of the line segment that is cut off by the first quadrant and is tangent to the curve $y = 3/x$ at some point?

42. What is the smallest possible area of the triangle that is cut off by the first quadrant and whose hypotenuse is tangent to the parabola $y = 4 - x^2$ at some point?

43. (a) If $C(x)$ is the cost of producing x units of a commodity, then the **average cost** per unit is $c(x) = C(x)/x$. Show that if the average cost is a minimum, then the marginal cost equals the average cost.

(b) If $C(x) = 16{,}000 + 200x + 4x^{3/2}$, in dollars, find (i) the cost, average cost, and marginal cost at a production level of 1000 units; (ii) the production level that will minimize the average cost; and (iii) the minimum average cost.

44. (a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.

(b) If $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$ is the cost function and $p(x) = 1700 - 7x$ is the demand function, find the production level that will maximize profit.

45. A baseball team plays in a stadium that holds 55,000 spectators. With ticket prices at \$10, the average attendance had been 27,000. When ticket prices were lowered to \$8, the average attendance rose to 33,000.

(a) Find the demand function, assuming that it is linear.

(b) How should ticket prices be set to maximize revenue?

46. During the summer months Terry makes and sells necklaces on the beach. Last summer he sold the necklaces for \$10 each and his sales averaged 20 per day. When he increased the price by \$1, he found that the average decreased by two sales per day.

(a) Find the demand function, assuming that it is linear.

(b) If the material for each necklace costs Terry \$6, what should the selling price be to maximize his profit?

47. A manufacturer has been selling 1000 television sets a week at \$450 each. A market survey indicates that for each \$10 rebate offered to the buyer, the number of sets sold will increase by 100 per week.

(a) Find the demand function.

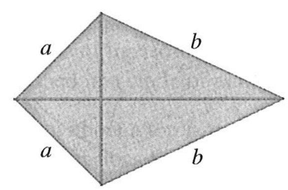
(b) How large a rebate should the company offer the buyer in order to maximize its revenue?

(c) If its weekly cost function is $C(x) = 68{,}000 + 150x$, how should the manufacturer set the size of the rebate in order to maximize its profit?

48. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?

49. Let a and b be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point (a, b).

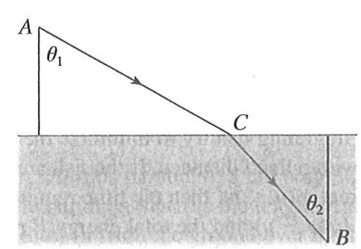
CAS **50.** The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?



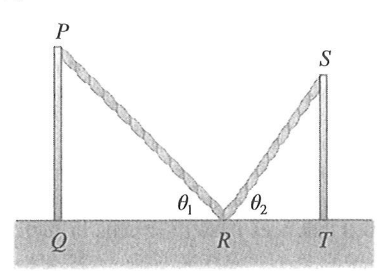
51. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

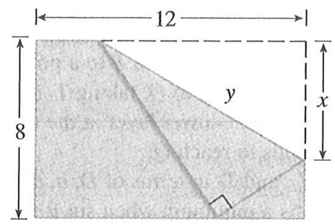
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.



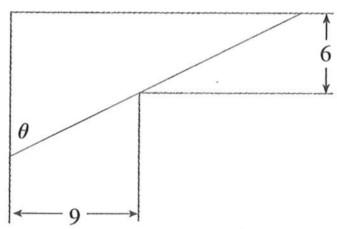
52. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.



53. The upper right-hand corner of a piece of paper, 12 in. by 8 in., as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?

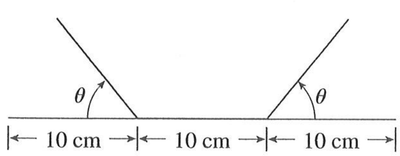


54. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?

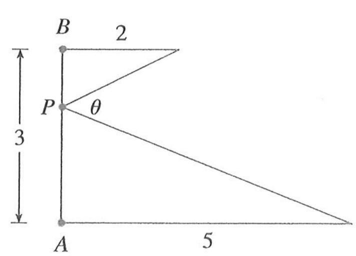


55. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W. [*Hint:* Express the area as a function of an angle θ.]

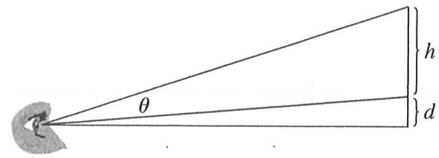
56. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?



57. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



58. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)



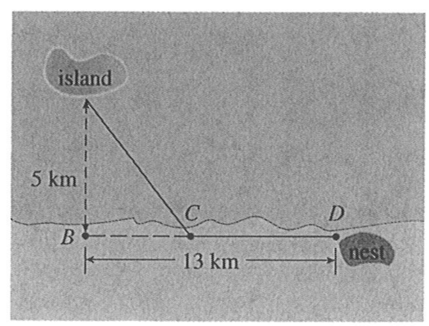
59. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than over land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

(a) In general, if it takes 1.4 times as much energy to fly over water as it does over land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.

(c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shore to D?

(d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than over land?

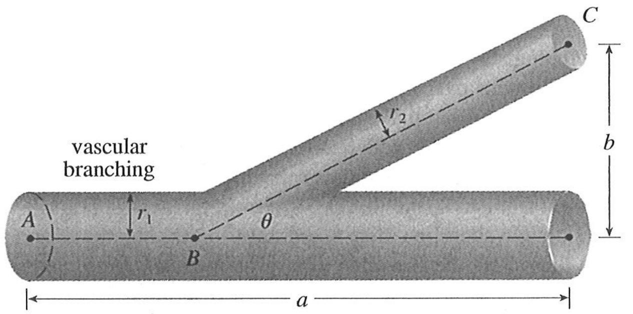


60. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's

Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally, but it also follows from Equation 6.7.2.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2.



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

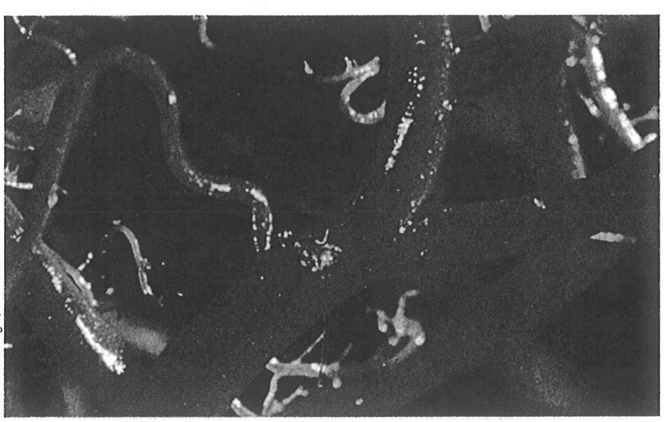
$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where a and b are the distances shown in the figure.

(b) Prove that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(c) Find the optimal branching angle (correct to the nearest degree) when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.

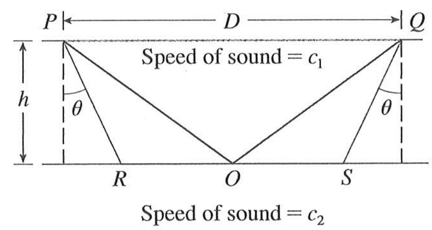


61. The speeds of sound c_1 in an upper layer and c_2 in a lower layer of rock and the thickness h of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. A dynamite charge is detonated at a point P and the transmitted signals are recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes T_1 seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking T_2 seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes T_3 seconds to reach Q.

(a) Express T_1, T_2, and T_3 in terms of D, h, c_1, c_2, and θ.

(b) Show that T_2 is a minimum when $\sin\theta = c_1/c_2$.

(c) Suppose that $D = 1$ km, $T_1 = 0.26$ s, $T_2 = 0.32$ s, and $T_3 = 0.34$ s. Find c_1, c_2, and h.



Note: Geophysicists use this technique when studying the structure of the earth's crust, whether searching for oil or examining fault lines.

62. Two light sources of identical strength are placed 10 m apart. An object is to be placed at a point P on a line ℓ parallel to the line joining the light sources and at a distance d meters from it (see the figure). We want to locate P on ℓ so that the intensity of illumination is minimized. We need to use the fact that the intensity of illumination for a single source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

(a) Find an expression for the intensity $I(x)$ at the point P.

(b) If $d = 5$ m, use graphs of $I(x)$ and $I'(x)$ to show that the intensity is minimized when $x = 5$ m, that is, when P is at the midpoint of ℓ.

(c) If $d = 10$ m, show that the intensity (perhaps surprisingly) is *not* minimized at the midpoint.

(d) Somewhere between $d = 5$ m and $d = 10$ m there is a transitional value of d at which the point of minimal illumination abruptly changes. Estimate this value of d by graphical methods. Then find the exact value of d.