37–40 Find the limit.

37. $\lim_{x \to -1^+} \sin^{-1} x$

38. $\lim_{x \to \infty} \arccos\left(\dfrac{1 + x^2}{1 + 2x^2}\right)$

39. $\lim_{x \to \infty} \arctan(e^x)$

40. $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

---

41. (a) Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

(b) If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

42. (a) Show that $f(x) = 2x + \cos x$ is one-to-one.
(b) What is the value of $f^{-1}(1)$?
(c) Use the formula from Exercise 41(a) to find $(f^{-1})'(1)$.

43. Use the formula from Exercise 41(a) to prove
(a) Formula 1 (b) Formula 4

44. (a) Sketch the graph of the function $f(x) = \sin(\sin^{-1} x)$.
(b) Sketch the graph of the function $g(x) = \sin^{-1}(\sin x)$, $x \in \mathbb{R}$.
(c) Show that $g'(x) = \dfrac{\cos x}{|\cos x|}$.
(d) Sketch the graph of $h(x) = \cos^{-1}(\sin x)$, $x \in \mathbb{R}$, and find its derivative.

3.7 Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_a x$ and, in particular, the natural logarithmic function $y = \ln x$. (It can be proved that logarithmic functions are differentiable; this is certainly plausible from their graphs. See Figure 4 in Section 1.6 for the graphs of the logarithmic functions.)

1
$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

PROOF Let $y = \log_a x$. Then

$$a^y = x$$

Differentiating this equation implicitly with respect to x, using Formula 3.4.5, we get

Formula 3.4.5 says that

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$a^y(\ln a)\frac{dy}{dx} = 1$$

and so
$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$ □

If we put $a = e$ in Formula 1, then the factor $\ln a$ on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $\log_e x = \ln x$:

2
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

By comparing Formulas 1 and 2, we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: The differentiation formula is simplest when $a = e$ because $\ln e = 1$.

V EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

In general, if we combine Formula 2 with the Chain Rule as in Example 1, we get

3 $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad \text{or} \qquad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE 2 Find $\dfrac{d}{dx}\ln(\sin x)$.

SOLUTION Using (3), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

EXAMPLE 4 Differentiating a logarithm with base 10 Differentiate $f(x) = \log_{10}(2 + \sin x)$.

SOLUTION Using Formula 1 with $a = 10$, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\log_{10}(2 + \sin x) \\ &= \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x) \\ &= \frac{\cos x}{(2 + \sin x)\ln 10} \end{aligned}$$

Figure 1 shows the graph of the function f of Example 5 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is rapidly decreasing.

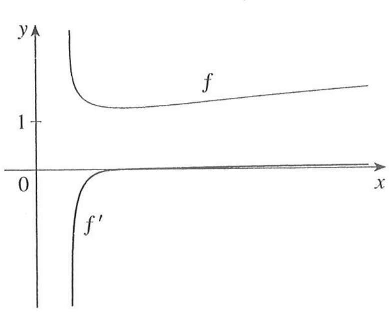

FIGURE 1

EXAMPLE 5 Simplifying before differentiating Find $\dfrac{d}{dx}\ln\dfrac{x + 1}{\sqrt{x - 2}}$.

SOLUTION 1

$$\begin{aligned} \frac{d}{dx}\ln\frac{x + 1}{\sqrt{x - 2}} &= \frac{1}{\dfrac{x + 1}{\sqrt{x - 2}}}\frac{d}{dx}\frac{x + 1}{\sqrt{x - 2}} \\ &= \frac{\sqrt{x - 2}}{x + 1}\,\frac{\sqrt{x - 2} \cdot 1 - (x + 1)\left(\tfrac{1}{2}\right)(x - 2)^{-1/2}}{x - 2} \\ &= \frac{x - 2 - \tfrac{1}{2}(x + 1)}{(x + 1)(x - 2)} \\ &= \frac{x - 5}{2(x + 1)(x - 2)} \end{aligned}$$

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right]$$

$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.)

Figure 2 shows the graph of the function $f(x) = \ln|x|$ in Example 6 and its derivative $f'(x) = 1/x$. Notice that when x is small, the graph of $y = \ln|x|$ is steep and so $f'(x)$ is large (positive or negative).

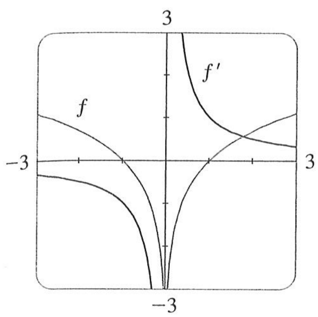

FIGURE 2

V EXAMPLE 6 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

The result of Example 6 is worth remembering:

4

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 7 Logarithmic differentiation Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

If we hadn't used logarithmic differentiation in Example 7, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can write $|y| = |f(x)|$ and use Equation 4. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 3.1.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

PROOF Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

Therefore

$$\frac{y'}{y} = \frac{n}{x}$$

Hence

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

⊘ You should distinguish carefully between the Power Rule $[(x^n)' = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(a^x)' = a^x \ln a]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

1. $\frac{d}{dx}(a^b) = 0$ $\quad$ (a and b are constants)

2. $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$

3. $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

V EXAMPLE 8 **What to do if both base and exponent contain x** Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Using logarithmic differentiation, we have

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x) \frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x} \ln x}\right) = e^{\sqrt{x} \ln x} \frac{d}{dx}\left(\sqrt{x} \ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$

Figure 3 illustrates Example 8 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

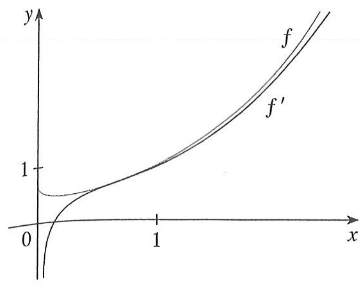

FIGURE 3

The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1 + x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x)$$

$$= \lim_{x \to 0} \ln(1 + x)^{1/x}$$

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1 + x)^{1/x} = 1$$

Then, by Theorem 2.4.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1 + x)^{1/x}$$

5
$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

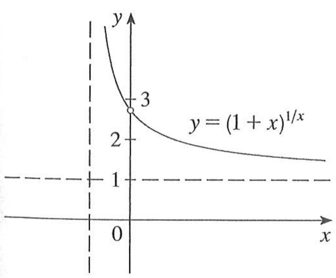

FIGURE 4

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

Formula 5 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 4 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

If we put $n = 1/x$ in Formula 5, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

6 $$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

3.7 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–20 Differentiate the function.

2. $f(x) = x \ln x - x$
3. $f(x) = \sin(\ln x)$
4. $f(x) = \ln(\sin^2 x)$
5. $f(x) = \log_2(1 - 3x)$
6. $f(x) = \log_5(xe^x)$
7. $f(x) = \sqrt[5]{\ln x}$
8. $f(x) = \ln \sqrt[5]{x}$
9. $f(x) = \sin x \ln(5x)$
10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$
11. $F(t) = \ln \dfrac{(2t + 1)^3}{(3t - 1)^4}$
12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$
13. $g(x) = \ln\left(x\sqrt{x^2 - 1}\right)$
14. $F(y) = y \ln(1 + e^y)$
15. $y = \ln|2 - x - 5x^2|$
16. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$
17. $y = \ln(e^{-x} + xe^{-x})$
18. $y = [\ln(1 + e^x)]^2$
19. $y = 2x \log_{10} \sqrt{x}$
20. $y = \log_2(e^{-x} \cos \pi x)$

21–22 Find y' and y''.

21. $y = x^2 \ln(2x)$
22. $y = \dfrac{\ln x}{x^2}$

23–24 Differentiate f and find the domain of f.

23. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$
24. $f(x) = \ln \ln \ln x$

25–27 Find an equation of the tangent line to the curve at the given point.

25. $y = \ln(x^2 - 3x + 1)$, $(3, 0)$
26. $y = \ln(x^3 - 7)$, $(2, 0)$
27. $y = \ln\left(xe^{x^2}\right)$, $(1, 1)$

28. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

29. (a) On what interval is $f(x) = x \ln x$ decreasing?
(b) On what interval is f concave upward?

30. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

31. Let $f(x) = cx + \ln(\cos x)$. For what value of c is $f'(\pi/4) = 6$?

32. Let $f(x) = \log_a(3x^2 - 2)$. For what value of a is $f'(1) = 3$?

33–42 Use logarithmic differentiation to find the derivative of the function.

33. $y = (2x + 1)^5(x^4 - 3)^6$
34. $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$
35. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$
36. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$
37. $y = x^x$
38. $y = x^{\cos x}$
39. $y = (\cos x)^x$
40. $y = \sqrt{x}^{\,x}$
41. $y = (\tan x)^{1/x}$
42. $y = (\sin x)^{\ln x}$

43. Find y' if $y = \ln(x^2 + y^2)$.

44. Find y' if $x^y = y^x$.

45. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

46. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

47. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

48. Show that $\displaystyle \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

Graphing calculator or computer with graphing software required

1. Homework Hints available in TEC