**70.** A tangent line is drawn to the hyperbola $xy = c$ at a point P.
  (a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P.
  (b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.

71. Evaluate $\lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$.

72. Draw a diagram showing two perpendicular lines that intersect on the y-axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

73. If $c > \frac{1}{2}$, how many lines through the point $(0, c)$ are normal lines to the parabola $y = x^2$? What if $c \leqslant \frac{1}{2}$?

74. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find its equation. If not, why not?
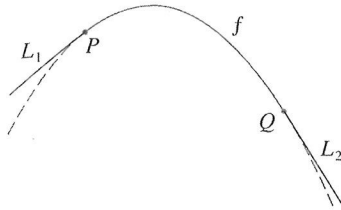
APPLIED PROJECT | **Building a Better Roller Coaster**

Suppose you are asked to design the first ascent and drop for a new roller coaster. By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6. You decide to connect these two straight stretches $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at the transition points P and Q. (See the figure.) To simplify the equations, you decide to place the origin at P.

1. (a) Suppose the horizontal distance between P and Q is 100 ft. Write equations in a, b, and c that will ensure that the track is smooth at the transition points.
   (b) Solve the equations in part (a) for a, b, and c to find a formula for $f(x)$.
   (c) Plot L_1, f, and L_2 to verify graphically that the transitions are smooth.
   (d) Find the difference in elevation between P and Q.

2. The solution in Problem 1 might *look* smooth, but it might not *feel* smooth because the piecewise defined function [consisting of $L_1(x)$ for $x < 0$, $f(x)$ for $0 \leqslant x \leqslant 100$, and $L_2(x)$ for $x > 100$] doesn't have a continuous second derivative. So you decide to improve the design by using a quadratic function $q(x) = ax^2 + bx + c$ only on the interval $10 \leqslant x \leqslant 90$ and connecting it to the linear functions by means of two cubic functions:

$$g(x) = kx^3 + lx^2 + mx + n \qquad 0 \leqslant x < 10$$

$$h(x) = px^3 + qx^2 + rx + s \qquad 90 < x \leqslant 100$$

   (a) Write a system of equations in 11 unknowns that ensure that the functions and their first two derivatives agree at the transition points.
   (b) Solve the equations in part (a) with a computer algebra system to find formulas for $q(x)$, $g(x)$, and $h(x)$.
   (c) Plot L_1, g, q, h, and L_2, and compare with the plot in Problem 1(c).

Graphing calculator or computer with graphing software required

CAS   Computer algebra system required

3.2 The Product and Quotient Rules

The formulas of this section enable us to differentiate new functions formed from old functions by multiplication or division.

The Product Rule

By analogy with the Sum and Difference Rules, one might be tempted to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example. Let $f(x) = x$
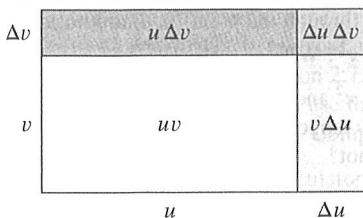
and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$. But $(fg)(x) = x^3$, so $(fg)'(x) = 3x^2$. Thus $(fg)' \neq f'g'$. The correct formula was discovered by Leibniz (soon after his false start) and is called the Product Rule.

Before stating the Product Rule, let's see how we might discover it. We start by assuming that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions. Then we can interpret the product uv as an area of a rectangle (see Figure 1). If x changes by an amount Δx, then the corresponding changes in u and v are

$$\Delta u = f(x + \Delta x) - f(x) \qquad \Delta v = g(x + \Delta x) - g(x)$$

and the new value of the product, $(u + \Delta u)(v + \Delta v)$, can be interpreted as the area of the large rectangle in Figure 1 (provided that Δu and Δv happen to be positive).

The change in the area of the rectangle is

1 $$\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v$$

$$= \text{the sum of the three shaded areas}$$

FIGURE 1
The geometry of the Product Rule

If we divide by Δx, we get

$$\frac{\Delta(uv)}{\Delta x} = u\,\frac{\Delta v}{\Delta x} + v\,\frac{\Delta u}{\Delta x} + \Delta u\,\frac{\Delta v}{\Delta x}$$

Recall that in Leibniz notation the definition of a derivative can be written as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If we now let $\Delta x \to 0$, we get the derivative of uv:

$$\frac{d}{dx}(uv) = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \to 0} \left(u\,\frac{\Delta v}{\Delta x} + v\,\frac{\Delta u}{\Delta x} + \Delta u\,\frac{\Delta v}{\Delta x} \right)$$

$$= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0} \Delta u \right)\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \right)$$

$$= u\,\frac{dv}{dx} + v\,\frac{du}{dx} + 0 \cdot \frac{dv}{dx}$$

2 $$\frac{d}{dx}(uv) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

(Notice that $\Delta u \to 0$ as $\Delta x \to 0$ since f is differentiable and therefore continuous.)

Although we started by assuming (for the geometric interpretation) that all the quantities are positive, we notice that Equation 1 is always true. (The algebra is valid whether u, v, Δu, and Δv are positive or negative.) So we have proved Equation 2, known as the Product Rule, for all differentiable functions u and v.

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\,\frac{d}{dx}\left[g(x)\right] + g(x)\,\frac{d}{dx}\left[f(x)\right]$$

In prime notation:

$$(fg)' = fg' + gf'$$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

EXAMPLE 1 **Using the Product Rule**
(a) If $f(x) = xe^x$, find $f'(x)$.
(b) Find the nth derivative, $f^{(n)}(x)$.

SOLUTION
(a) By the Product Rule, we have

$$f'(x) = \frac{d}{dx}(xe^x)$$

$$= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x \cdot 1 = (x + 1)e^x$$

(b) Using the Product Rule a second time, we get

$$f''(x) = \frac{d}{dx}[(x + 1)e^x]$$

$$= (x + 1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x + 1)$$

$$= (x + 1)e^x + e^x \cdot 1 = (x + 2)e^x$$

Further applications of the Product Rule give

$$f'''(x) = (x + 3)e^x \qquad f^{(4)}(x) = (x + 4)e^x$$

In fact, each successive differentiation adds another term e^x, so

$$f^{(n)}(x) = (x + n)e^x$$

Figure 2 shows the graphs of the function f of Example 1 and its derivative f'. Notice that $f'(x)$ is positive when f is increasing and negative when f is decreasing.
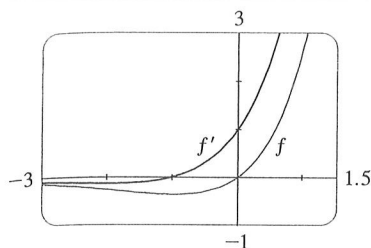

FIGURE 2

In Example 2, a and b are constants. It is customary in mathematics to use letters near the beginning of the alphabet to represent constants and letters near the end of the alphabet to represent variables.

EXAMPLE 2 **Differentiating a function with arbitrary constants**
Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

SOLUTION 1 Using the Product Rule, we have

$$f'(t) = \sqrt{t}\,\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}(\sqrt{t})$$

$$= \sqrt{t} \cdot b + (a + bt) \cdot \tfrac{1}{2}t^{-1/2}$$

$$= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}}$$

SOLUTION 2 If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$

$$f'(t) = \tfrac{1}{2}at^{-1/2} + \tfrac{3}{2}bt^{1/2}$$

which is equivalent to the answer given in Solution 1.

Example 2 shows that it is sometimes easier to simplify a product of functions before differentiating than to use the Product Rule. In Example 1, however, the Product Rule is the only possible method.

EXAMPLE 3 If $f(x) = \sqrt{x}\, g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$f'(x) = \frac{d}{dx}\left[\sqrt{x}\, g(x)\right] = \sqrt{x}\, \frac{d}{dx}\left[g(x)\right] + g(x)\, \frac{d}{dx}\left[\sqrt{x}\,\right]$$

$$= \sqrt{x}\, g'(x) + g(x) \cdot \tfrac{1}{2}x^{-1/2} = \sqrt{x}\, g'(x) + \frac{g(x)}{2\sqrt{x}}$$

So

$$f'(4) = \sqrt{4}\, g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = 6.5$$

☑ EXAMPLE 4 **Interpreting the terms in the Product Rule** A telephone company wants to estimate the number of new residential phone lines that it will need to install during the upcoming month. At the beginning of January the company had 100,000 subscribers, each of whom had 1.2 phone lines, on average. The company estimated that its subscribership was increasing at the rate of 1000 monthly. By polling its existing subscribers, the company found that each intended to install an average of 0.01 new phone lines by the end of January. Estimate the number of new lines the company will have to install in January by calculating the rate of increase of lines at the beginning of the month.

SOLUTION Let $s(t)$ be the number of subscribers and let $n(t)$ be the number of phone lines per subscriber at time t, where t is measured in months and $t = 0$ corresponds to the beginning of January. Then the total number of lines is given by

$$L(t) = s(t)n(t)$$

and we want to find $L'(0)$. According to the Product Rule, we have

$$L'(t) = \frac{d}{dt}\left[s(t)n(t)\right] = s(t)\, \frac{d}{dt}\, n(t) + n(t)\, \frac{d}{dt}\, s(t)$$

We are given that $s(0) = 100{,}000$ and $n(0) = 1.2$. The company's estimates concerning rates of increase are that $s'(0) \approx 1000$ and $n'(0) \approx 0.01$. Therefore

$$L'(0) = s(0)n'(0) + n(0)s'(0)$$

$$\approx 100{,}000 \cdot 0.01 + 1.2 \cdot 1000 = 2200$$

The company will need to install approximately 2200 new phone lines in January.

Notice that the two terms arising from the Product Rule come from different sources— old subscribers and new subscribers. One contribution to L' is the number of existing subscribers (100,000) times the rate at which they order new lines (about 0.01 per subscriber monthly). A second contribution is the average number of lines per subscriber (1.2 at the beginning of the month) times the rate of increase of subscribers (1000 monthly).

The Quotient Rule

We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in much the same way that we found the Product Rule. If x, u, and v change by amounts Δx, Δu, and Δv, then the corresponding change in the quotient u/v is

$$\Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)} = \frac{v\,\Delta u - u\,\Delta v}{v(v + \Delta v)}$$

so

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{\Delta x \to 0} \frac{\Delta(u/v)}{\Delta x} = \lim_{\Delta x \to 0} \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

As $\Delta x \to 0$, $\Delta v \to 0$ also, because $v = g(x)$ is differentiable and therefore continuous. Thus, using the Limit Laws, we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \displaystyle\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} - u \displaystyle\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x}}{v \displaystyle\lim_{\Delta x \to 0}(v + \Delta v)} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

In prime notation:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

The Quotient Rule and the other differentiation formulas enable us to compute the derivative of any rational function, as the next example illustrates.

We can use a graphing device to check that the answer to Example 5 is plausible. Figure 3 shows the graphs of the function of Example 5 and its derivative. Notice that when y grows rapidly (near -2), y' is large. And when y grows slowly, y' is near 0.
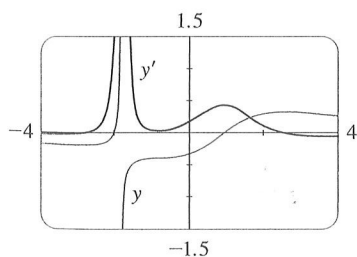

FIGURE 3

V **EXAMPLE 5** **Using the Quotient Rule** Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

V **EXAMPLE 6** Find an equation of the tangent line to the curve $y = e^x/(1 + x^2)$ at the point $\left(1, \frac{1}{2}e\right)$.

SOLUTION According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1 + x^2)\dfrac{d}{dx}(e^x) - e^x\dfrac{d}{dx}(1 + x^2)}{(1 + x^2)^2}$$

$$= \frac{(1 + x^2)e^x - e^x(2x)}{(1 + x^2)^2} = \frac{e^x(1 - x)^2}{(1 + x^2)^2}$$

FIGURE 4

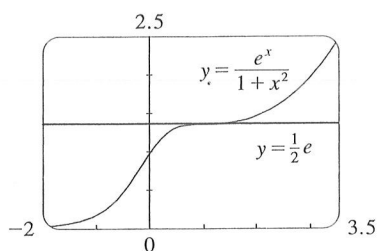So the slope of the tangent line at $\left(1, \frac{1}{2}e\right)$ is

$$\frac{dy}{dx}\bigg|_{x=1} = 0$$

This means that the tangent line at $\left(1, \frac{1}{2}e\right)$ is horizontal and its equation is $y = \frac{1}{2}e$. [See Figure 4. Notice that the function is increasing and crosses its tangent line at $\left(1, \frac{1}{2}e\right)$.]

Note: Don't use the Quotient Rule *every* time you see a quotient. Sometimes it's easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule, it is much easier to perform the division first and write the function as

$$F(x) = 3x + 2x^{-1/2}$$

before differentiating.

We summarize the differentiation formulas we have learned so far as follows.

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f + g)' = f' + g' \qquad (f - g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

3.2 Exercises

1. Find the derivative of $f(x) = (1 + 2x^2)(x - x^2)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

2. Find the derivative of the function

$$F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

3–24 Differentiate.

3. $f(x) = (x^3 + 2x)e^x$

4. $g(x) = \sqrt{x}\, e^x$

5. $y = \dfrac{e^x}{x^2}$

6. $y = \dfrac{e^x}{1 + x}$

7. $g(x) = \dfrac{3x - 1}{2x + 1}$

8. $f(t) = \dfrac{2t}{4 + t^2}$

9. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3)$

10. $R(t) = (t + e^t)(3 - \sqrt{t})$

11. $y = \dfrac{x^3}{1 - x^2}$

12. $y = \dfrac{x + 1}{x^3 + x - 2}$

13. $y = \dfrac{t^2 + 2}{t^4 - 3t^2 + 1}$

14. $y = \dfrac{t}{(t - 1)^2}$

⊞ Graphing calculator or computer with graphing software required   **1.** Homework Hints available in TEC

15. $y = (r^2 - 2r)e^r$

16. $y = \dfrac{1}{s + ke^s}$

17. $y = \dfrac{v^3 - 2v\sqrt{v}}{v}$

18. $z = w^{3/2}(w + ce^w)$

19. $f(t) = \dfrac{2t}{2 + \sqrt{t}}$

20. $g(t) = \dfrac{t - \sqrt{t}}{t^{1/3}}$

21. $f(x) = \dfrac{A}{B + Ce^x}$

22. $f(x) = \dfrac{1 - xe^x}{x + e^x}$

23. $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$

24. $f(x) = \dfrac{ax + b}{cx + d}$

25–28 Find $f'(x)$ and $f''(x)$.

25. $f(x) = x^4 e^x$

26. $f(x) = x^{5/2} e^x$

27. $f(x) = \dfrac{x^2}{1 + 2x}$

28. $f(x) = \dfrac{x}{x^2 - 1}$

29–30 Find an equation of the tangent line to the given curve at the specified point.

29. $y = \dfrac{2x}{x + 1}$, $\quad (1, 1)$

30. $y = \dfrac{e^x}{x}$, $\quad (1, e)$

31–32 Find equations of the tangent line and normal line to the given curve at the specified point.

31. $y = 2xe^x$, $\quad (0, 0)$

32. $y = \dfrac{\sqrt{x}}{x + 1}$, $\quad (4, 0.4)$

---

33. (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $\left(-1, \frac{1}{2}\right)$.
  (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

34. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point $(3, 0.3)$.
  (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

35. (a) If $f(x) = (x^3 - x)e^x$, find $f'(x)$.
  (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

36. (a) If $f(x) = e^x/(2x^2 + x + 1)$, find $f'(x)$.
  (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

37. (a) If $f(x) = (x^2 - 1)/(x^2 + 1)$, find $f'(x)$ and $f''(x)$.
  (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

38. (a) If $f(x) = (x^2 - 1)e^x$, find $f'(x)$ and $f''(x)$.
  (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

39. If $f(x) = x^2/(1 + x)$, find $f''(1)$.

40. If $g(x) = x/e^x$, find $g^{(n)}(x)$.

41. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.
  (a) $(fg)'(5)$ $\qquad$ (b) $(f/g)'(5)$
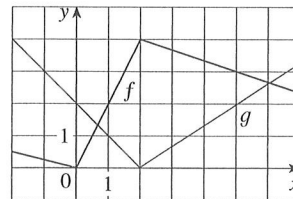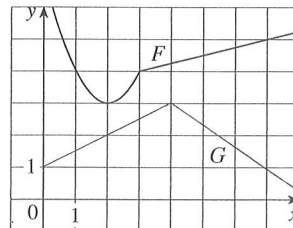 (c) $(g/f)'(5)$

42. Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$.
  (a) $h(x) = 5f(x) - 4g(x)$ $\qquad$ (b) $h(x) = f(x)g(x)$
  (c) $h(x) = \dfrac{f(x)}{g(x)}$ $\qquad$ (d) $h(x) = \dfrac{g(x)}{1 + f(x)}$

43. If $f(x) = e^x g(x)$, where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

44. If $h(2) = 4$ and $h'(2) = -3$, find
$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\Bigg|_{x=2}$$

45. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.
  (a) Find $u'(1)$. $\qquad$ (b) Find $v'(5)$.

46. Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where F and G are the functions whose graphs are shown.
  (a) Find $P'(2)$. $\qquad$ (b) Find $Q'(7)$.

47. If g is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = xg(x)$ (b) $y = \dfrac{x}{g(x)}$ (c) $y = \dfrac{g(x)}{x}$

48. If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2 f(x)$ (b) $y = \dfrac{f(x)}{x^2}$

(c) $y = \dfrac{x^2}{f(x)}$ (d) $y = \dfrac{1 + xf(x)}{\sqrt{x}}$

49. In this exercise we estimate the rate at which the total personal income is rising in the Richmond-Petersburg, Virginia, metropolitan area. In 1999, the population of this area was 961,400, and the population was increasing at roughly 9200 people per year. The average annual income was \$30,593 per capita, and this average was increasing at about \$1400 per year (a little above the national average of about \$1225 yearly). Use the Product Rule and these figures to estimate the rate at which total personal income was rising in the Richmond-Petersburg area in 1999. Explain the meaning of each term in the Product Rule.

50. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.
(a) What does it mean to say that $f(20) = 10,000$ and $f'(20) = -350$?
(b) Assuming the values in part (a), find $R'(20)$ and interpret your answer.

51. On what interval is the function $f(x) = x^3 e^x$ increasing?

52. On what interval is the function $f(x) = x^2 e^x$ concave downward?

53. How many tangent lines to the curve $y = x/(x + 1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

54. Find equations of the tangent lines to the curve

$$y = \frac{x - 1}{x + 1}$$

that are parallel to the line $x - 2y = 2$.

55. Find $R'(0)$, where

$$R(x) = \frac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9}$$

Hint: Instead of finding $R'(x)$ first, let $f(x)$ be the numerator and $g(x)$ the denominator of $R(x)$ and compute $R'(0)$ from $f(0)$, $f'(0)$, $g(0)$, and $g'(0)$.

56. Use the method of Exercise 55 to compute $Q'(0)$, where

$$Q(x) = \frac{1 + x + x^2 + xe^x}{1 - x + x^2 - xe^x}$$

57. (a) Use the Product Rule twice to prove that if f, g, and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.
(b) Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(c) Use part (b) to differentiate $y = e^{3x}$.

58. (a) If $F(x) = f(x)g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.
(b) Find similar formulas for F''' and $F^{(4)}$.
(c) Guess a formula for $F^{(n)}$.

59. Find expressions for the first five derivatives of $f(x) = x^2 e^x$. Do you see a pattern in these expressions? Guess a formula for $f^{(n)}(x)$ and prove it using mathematical induction.

60. (a) If g is differentiable, the **Reciprocal Rule** says that

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{[g(x)]^2}$$

Use the Quotient Rule to prove the Reciprocal Rule.
(b) Use the Reciprocal Rule to differentiate the function in Exercise 16.
(c) Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is,

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

for all positive integers n.

3.3 Derivatives of Trigonometric Functions

A review of the trigonometric functions is given in Appendix C.

Before starting this section, you might need to review the trigonometric functions. In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x. A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot. Recall