Homework 31 Solutions Math 141

18. Let x = length of the north side. Let y = length of the west side.
Cost = $2x(10) + 2y(15) = 20x + 30y = 480$
We want to maximize xy subject to $20x + 30y - 480 = 0$.

$$F(x, y, \lambda) = xy + \lambda(20x + 30y - 480)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = y + 20\lambda = 0 \\ \dfrac{\partial F}{\partial y} = x + 30\lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = 20x + 30y - 480 = 0 \end{array}\right\} \left.\begin{array}{c} \lambda = \dfrac{-y}{20} \\ \lambda = \dfrac{-x}{30} \\ 2x + 3y = 48 \end{array}\right\} \left.\begin{array}{l} 2x - 3y = 0 \\ 2x + 3y = 48 \end{array}\right\} \begin{array}{l} x = 12 \\ y = 8 \end{array}$$

The dimensions of the garden should be 12 ft × 8 ft.

19. Let x = length of a side of the base. Let y = height of the box.
Area = $x^2 + 4xy = 300$

Maximize the volume = x^2y subject to $x^2 + 4xy - 300 = 0$.

$$F(x, y, \lambda) = x^2y + \lambda(x^2 + 4xy - 300)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = 2xy + 2x\lambda + 4y\lambda = 0 \\ \dfrac{\partial F}{\partial y} = x^2 + 4x\lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x^2 + 4xy - 300 = 0 \end{array}\right\} \left.\begin{array}{c} \lambda = \dfrac{-2xy}{2(x+2y)} \\ \lambda = \dfrac{-x}{4x} = \dfrac{-x}{4} \\ x^2 + 4xy = 300 \end{array}\right\} \left.\begin{array}{l} x - 2y = 0 \\ x^2 + 4xy = 300 \end{array}\right\} \begin{array}{l} x = 10 \\ y = 5 \end{array}$$

The sides of the base should be 10 in. and the height should be 5 in.

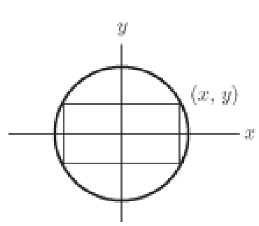
20. Let x = the number of units of labor and let y = the number of units of capital cost. The problem is to minimize $1000\sqrt{6x^2 + y^2}$ subject to $5000 - 480x - 40y = 0$.

$$F(x, y, \lambda) = 1000\sqrt{6x^2 + y^2} + \lambda(5000 - 480x - 40y)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = \dfrac{6000x}{\sqrt{6x^2 + y^2}} - 480\lambda = 0 \\ \dfrac{\partial F}{\partial y} = \dfrac{1000y}{\sqrt{6x^2 + y^2}} - 40\lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = 5000 - 480x - 40y = 0 \end{array}\right\} \left.\begin{array}{c} \lambda = \dfrac{12.5x}{\sqrt{6x^2 + y^2}} \\ \lambda = \dfrac{25y}{\sqrt{6x^2 + y^2}} \\ 48x + 4y = 500 \end{array}\right\} \left.\begin{array}{l} x = 2y \\ 100y = 500 \end{array}\right\} \begin{array}{l} x = 10 \\ y = 5 \end{array}$$

There should be 10 units of labor and 5 units of capital to minimize the amount of space required.

21.



The length of the rectangle is $2x$, and the width of the rectangle is $2y$.
The problem is to maximize $4xy$ subject to $1 - x^2 - y^2 = 0$.

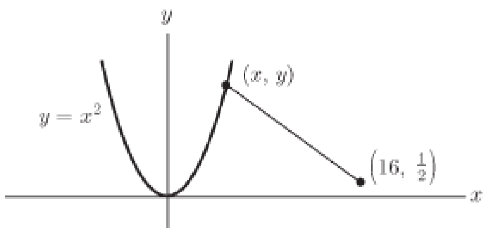
$$F(x, y, \lambda) = 4xy + \lambda(1 - x^2 - y^2)$$

(*continued*)

$$\left.\begin{aligned}\frac{\partial F}{\partial x}&=4y-2\lambda x=0\\ \frac{\partial F}{\partial y}&=4x-2\lambda y=0\\ \frac{\partial F}{\partial \lambda}&=1-x^2-y^2=0\end{aligned}\right\}\left.\begin{aligned}\lambda&=\frac{2y}{x}\\ \lambda&=\frac{2x}{y}\\ x^2+y^2&=1\end{aligned}\right\}\left.\begin{aligned}&\text{Assuming } x\neq 0,\ y\neq 0\\ &\frac{2y}{x}=\frac{2x}{y}\\ &x^2+y^2=1\end{aligned}\right\}\left.\begin{aligned}x^2&=y^2\\ 2x^2&=1\end{aligned}\right\}\begin{aligned}&\text{Assuming } x>0,\ y>0,\\ &x=\frac{\sqrt{2}}{2}\text{ and } y=\frac{\sqrt{2}}{2}\end{aligned}$$

The dimensions of the rectangle are $\sqrt{2}\times\sqrt{2}$.

22.



Following the hint, the problem is to minimize $(x-16)^2+\left(y-\frac{1}{2}\right)^2$ subject to $y-x^2=0$.

$$F(x, y, \lambda)=(x-16)^2+\left(y-\frac{1}{2}\right)^2+\lambda(y-x^2)$$

$$\left.\begin{aligned}\frac{\partial F}{\partial x}&=2x-32-2\lambda x=0\\ \frac{\partial F}{\partial y}&=2y-1+\lambda=0\\ \frac{\partial F}{\partial \lambda}&=y-x^2=0\end{aligned}\right\}\left.\begin{aligned}\lambda&=1-\frac{16}{x}\\ \lambda&=1-2y\\ y&=x^2\end{aligned}\right\}\left.\begin{aligned}&x=0,\ y=0\text{ or}\\ &\frac{16}{x}=2y\\ &y=x^2\end{aligned}\right\}\begin{aligned}&x=0,\ y=0\\ &\text{or}\\ &x=2,\ y=4\end{aligned}$$

To decide which of (0, 0) or (2, 4) is the closer point, check that $(0-16)^2+\left(0-\frac{1}{2}\right)^2>(2-16)^2+\left(4-\frac{1}{2}\right)^2$.

Thus (2, 4) is the desired point.

23. The problem is to maximize $3x + 4y$ subject to $18,000-9x^2-4y^2=0$, $x\geq 0, y\geq 0$.

$$F(x, y, \lambda)=3x+4y+\lambda(18,000-9x^2-4y^2)$$

$$\left.\begin{aligned}\frac{\partial F}{\partial x}&=3-18\lambda x=0\\ \frac{\partial F}{\partial y}&=4-8\lambda x=0\\ \frac{\partial F}{\partial \lambda}&=18,000-9x^2-4y^2=0\end{aligned}\right\}\left.\begin{aligned}\lambda&=\frac{1}{6x}\\ \lambda&=\frac{1}{2y}\\ 9x^2+4y^2&=18,000\end{aligned}\right\}\left.\begin{aligned}&\text{If } x\neq 0 \text{ and } y\neq 0 \text{ then}\\ &y=3x.\\ &9x^2+36x^2=18,000\end{aligned}\right\}\begin{aligned}x&=20\\ y&=60\end{aligned}$$

Technically, we should also check the solutions $x=0,\ y=\sqrt{\frac{18,000}{9}}\approx 44.7$ and $y=0, x=\sqrt{\frac{18,000}{36}}\approx 22.4$.

These both give smaller values in the objective function $3x + 4y$ than does (20, 60).

24. We want to minimize the function $P = 2x + 10y$ subject to the constraint $4x^2 + 25y^2 = 50,000$.

$$F(x, y, \lambda) = 2x + 10y + \lambda(4x^2 + 25y^2 - 50,000)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2 + 8\lambda x = 0 \\ \frac{\partial F}{\partial y} &= 10 + 50\lambda y = 0 \\ \frac{\partial F}{\partial \lambda} &= 4x^2 + 25y^2 - 50,000 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= \frac{-1}{4x} \\ \lambda &= \frac{-1}{5y} \\ 4x^2 + 25y^2 &= 50,000 \end{aligned}\right\} \begin{aligned} x &= \frac{5}{4}y \\ 4x^2 + 25y^2 &= 50,000 \end{aligned}$$

Solving gives $\frac{125}{4}y^2 = 50,000$ or $y = 40$, hence $x = 50$.

They should produce 50 units of product *A* and 40 units of product *B*.

25. **a.** $F(x, y, \lambda) = 96x + 162y + \lambda\left(3456 - 64x^{3/4}y^{1/4}\right)$

Note that $3456 = 64x^{3/4}y^{1/4}$ implies $x \neq 0, y \neq 0$.

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 96 - 48\lambda x^{-1/4}y^{1/4} \\ \frac{\partial F}{\partial y} &= 162 - 16\lambda x^{3/4}y^{-3/4} \\ \frac{\partial F}{\partial \lambda} &= 3456 - 64x^{3/4}y^{1/4} \end{aligned}\right\} \left.\begin{aligned} \lambda &= 2x^{1/4}y^{-1/4} \\ \lambda &= \frac{81}{8}x^{-3/4}y^{3/4} \\ 3456 &= 64x^{3/4}y^{1/4} \end{aligned}\right\} \text{Dividing 1st equation by 2nd gives} \left.\begin{aligned} \frac{16}{81}xy^{-1} &= 1 \\ y &= \frac{16}{81}x \\ 3456 &= 64x^{3/4}\left(\frac{16}{81}x\right)^{1/4} \end{aligned}\right\} \begin{aligned} x &= 81 \\ y &= 16 \end{aligned}$$

b. $\lambda = 2(81)^{1/4}(16)^{-1/4} = 3$

c. The production function is $f(x, y) = 64x^{3/4}y^{1/4}$. Thus

$$\frac{\text{marginal productivity of labor}}{\text{marginal productivity of capital}} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{48x^{-1/4}y^{1/4}}{16x^{3/4}y^{-3/4}} = \frac{48y}{16x}.$$

When $x = 81$ and $y = 16$, $\frac{48y}{16x} = \frac{48 \cdot 16}{16 \cdot 81} = \frac{96}{162}$, which is the ratio of the unit cost of labor and capital.

26. $F(x, y, \lambda) = 94x - \frac{x^2}{10} + 80y - \frac{y^2}{20} - 20,000 + \lambda\left(14 - \frac{x}{10} + \frac{y}{20}\right)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 94 - \frac{x}{5} - \frac{\lambda}{10} = 0 \\ \frac{\partial F}{\partial y} &= 80 - \frac{y}{10} + \frac{\lambda}{20} = 0 \\ \frac{\partial F}{\partial \lambda} &= 14 - \frac{x}{10} + \frac{y}{20} = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= 940 - 2x \\ \lambda &= -1600 - 2x \\ \frac{x}{10} - \frac{y}{20} &= 14 \end{aligned}\right\} \left.\begin{aligned} 2x + 2y &= 2540 \\ 2x - y &= 280 \end{aligned}\right\} \begin{aligned} x &= \frac{1550}{3} \\ y &= \frac{2260}{3} \end{aligned}$$

27. $F(x, y, z, \lambda) = xyz + \lambda(36 - x - 6y - 3z)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= yz - \lambda = 0 \\ \frac{\partial F}{\partial y} &= xz - 6\lambda = 0 \\ \frac{\partial F}{\partial z} &= xy - 3\lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= 36 - x - 6y - 3z = 0 \end{aligned}\right\} \left.\begin{aligned} &\lambda = yz \\ &\lambda = \frac{xz}{6} \\ &\lambda = \frac{xy}{3} \\ &x + 6y + 3z = 36 \end{aligned}\right\} \left.\begin{aligned} &y = \frac{x}{6} \\ &\frac{z}{6} = \frac{y}{3} \\ &x + 6y + 3z = 36 \end{aligned}\right\} \left.\begin{aligned} &x = 6y \\ &3z = 6y \\ &3(6y) = 36 \end{aligned}\right\} \begin{aligned} &x = 12 \\ &y = 2 \\ &z = 4 \end{aligned}$$

28. $F(x, y, z, \lambda) = xy + 3xz + 3yz + \lambda(9 - xyz)$

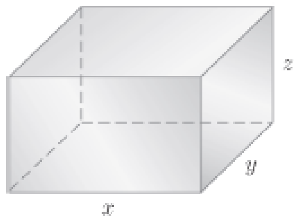
$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= y + 3z - \lambda yz = 0 \\ \frac{\partial F}{\partial y} &= x + 3z - \lambda xz = 0 \\ \frac{\partial F}{\partial z} &= 3x + 3y - \lambda xy = 0 \\ \frac{\partial F}{\partial \lambda} &= 9 - xyz = 0 \end{aligned}\right\} \left.\begin{aligned} &\lambda = \frac{1}{z} + \frac{3}{y} \\ &\lambda = \frac{1}{z} + \frac{3}{x} \\ &\lambda = \frac{3}{y} + \frac{3}{x} \\ &xyz = 9 \end{aligned}\right\} \left.\begin{aligned} &\frac{3}{y} = \frac{3}{x} \\ &\frac{1}{z} = \frac{3}{y} \\ &xyz = 9 \end{aligned}\right\} \left.\begin{aligned} &x = y \\ &z = \frac{y}{3} \\ &yy\left(\frac{y}{3}\right) = 9 \text{ or } y^3 = 27 \text{ or } y = 3 \end{aligned}\right\} \begin{aligned} &x = 3 \\ &y = 3 \\ &z = 1 \end{aligned}$$

29. $F(x, y, z, \lambda) = 3x + 5y + z - x^2 - y^2 - z^2 + \lambda(6 - x - y - z)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 3 - 2x - \lambda = 0 \\ \frac{\partial F}{\partial y} &= 5 - 2y - \lambda = 0 \\ \frac{\partial F}{\partial z} &= 1 - 2z - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= 6 - x - y - z = 0 \end{aligned}\right\} \left.\begin{aligned} &\lambda = 3 - 2x \\ &\lambda = 5 - 2y \\ &\lambda = 1 - 2z \\ &x + y + z = 6 \end{aligned}\right\} \left.\begin{aligned} &-2x + 2y = 2 \\ &-2y + 2z = -4 \\ &x + y + z = 6 \end{aligned}\right\} \left.\begin{aligned} &x = -1 + y \\ &z = -2 + y \\ &(-1 + y) + y + (-2 + y) = 6 \end{aligned}\right\} \begin{aligned} &x = 2 \\ &y = 3 \\ &z = 1 \end{aligned}$$

30. $F(x, y, z, \lambda) = x^2 + y^2 + z^2 - 3x - 5y - z + \lambda(20 - 2x - y - z)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2x - 3 - 2\lambda = 0 \\ \frac{\partial F}{\partial y} &= 2y - 5 - \lambda = 0 \\ \frac{\partial F}{\partial z} &= 2z - 1 - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= 20 - 2x - y - z = 0 \end{aligned}\right\} \left.\begin{aligned} &\lambda = x - \frac{3}{2} \\ &\lambda = 2y - 5 \\ &\lambda = 2z - 1 \\ &2x + y + z = 20 \end{aligned}\right\} \left.\begin{aligned} &x - 2y = -\frac{7}{2} \\ &2y - 2z = 4 \\ &2x + y + z = 20 \end{aligned}\right\} \left.\begin{aligned} &x = 2y - \frac{7}{2} \\ &z = -2 + y \\ &2\left(2y - \frac{7}{2}\right) + y + y - 2 = 20 \end{aligned}\right\} \begin{aligned} &x = \frac{37}{6} \\ &y = \frac{29}{6} \\ &z = \frac{17}{6} \end{aligned}$$

31. 

The problem is to minimize $3xy + 2xz + 2yz$ subject to the constraint $xyz = 12$ or $xyz - 12 = 0$.
$F(x, y, z, \lambda) = 3xy + 2xz + 2yz + \lambda(12 - xyz)$
(Note that $xyz = 12$ implies that $x \neq 0$, $y \neq 0$, $z \neq 0$.)

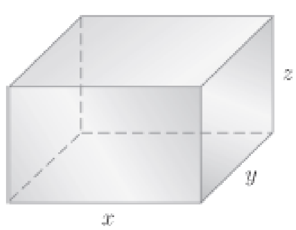
$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = 3y + 2z - \lambda yz = 0 \\ \frac{\partial F}{\partial y} = 3x + 2z - \lambda xz = 0 \\ \frac{\partial F}{\partial z} = 2x + 2y - \lambda xy = 0 \\ \frac{\partial F}{\partial \lambda} = 12 - xyz = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = \frac{3}{z} + \frac{2}{y} \\ \lambda = \frac{3}{z} + \frac{2}{x} \\ \lambda = \frac{2}{y} + \frac{2}{x} \\ xyz = 12 \end{array}\right\} \left.\begin{array}{l} x = y \\ z = \frac{3}{2}y \\ y^2\left(\frac{3}{2}y\right) = 12 \end{array}\right\} \begin{array}{l} x = 2 \\ y = 2 \\ z = 3 \end{array}$$

The box is 2 ft × 2 ft × 3 ft.

32. The problem is to maximize xyz subject to $x + y + z - 15 = 0$, $x > 0$, $y > 0$, $z > 0$.
$F(x, y, z, \lambda) = xyz + \lambda(x + y + z - 15)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = yz + \lambda = 0 \\ \frac{\partial F}{\partial y} = xz + \lambda = 0 \\ \frac{\partial F}{\partial z} = xy + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = x + y + z - 15 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -yz \\ \lambda = -xz \\ \lambda = -xy \\ x + y + z = 15 \end{array}\right\} \left.\begin{array}{l} x = y \\ z = y \\ 3y = 15 \end{array}\right\} \begin{array}{l} x = 5 \\ y = 5 \\ z = 5 \end{array}$$

The numbers are 5, 5 and 5.

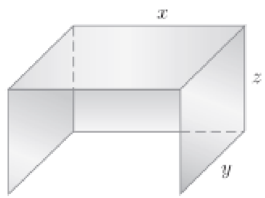
33. 

Let x, y, z be as shown in the figure. The problem is to minimize $xy + 2xz + 2yz$ subject to $xyz = 32$ or $32 - xyz = 0$.

$F(x, y, z, \lambda) = xy + 2xz + 2yz + \lambda(32 - xyz)$
(Note that $xyz = 32$ implies $x \neq 0$, $y \neq 0$, $z \neq 0$.)

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = y + 2z - \lambda yz = 0 \\ \frac{\partial F}{\partial y} = x + 2z - \lambda xz = 0 \\ \frac{\partial F}{\partial z} = 2x + 2y - \lambda xy = 0 \\ \frac{\partial F}{\partial \lambda} = xyz - 32 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = \frac{1}{z} + \frac{2}{y} \\ \lambda = \frac{1}{z} + \frac{2}{x} \\ \lambda = \frac{2}{y} + \frac{2}{x} \\ xyz = 32 \end{array}\right\} \text{Equating expressions for } \lambda \text{ gives} \left.\begin{array}{l} x = y \\ z = \frac{1}{2}y \\ \frac{1}{2}y^3 = 32 \end{array}\right\} \begin{array}{l} x = 4 \\ y = 4 \\ z = 2 \end{array}$$

The dimensions of the tank are 4 ft × 4 ft × 2 ft.

34. 

Let x, y, z be as shown in the figure. The problem is to maximize xyz subje to $xy + xz + 2yz = 96$, $x > 0$, $y > 0$, $z > 0$.

$F(x, y, z, \lambda) = xyz + \lambda(xy + xz + 2yz - 96)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= yz + \lambda(y+z) = 0 \\ \frac{\partial F}{\partial y} &= xz + \lambda(x+2z) = 0 \\ \frac{\partial F}{\partial z} &= xy + \lambda(x+2y) = 0 \\ \frac{\partial F}{\partial \lambda} &= xy + xz + 2yz - 96 = 0 \end{aligned}\right\} \begin{aligned} \lambda &= -\frac{yz}{y+z} \\ \lambda &= -\frac{xz}{x+2z} \\ \lambda &= -\frac{xy}{x+2y} \\ xy + xz &+ 2yz = 96 \end{aligned} \Bigg\} \text{Equating expressions for } \lambda \text{ gives } \left.\begin{aligned} x &= 2y \\ z &= y \\ 2y^2 + 2y^2 + 2y^2 &= 96 \end{aligned}\right\} \begin{aligned} x &= 8 \\ y &= 4 \\ z &= 4 \end{aligned}$$

The dimensions of the shelter are 8 ft $\times$ 4 ft $\times$ 4 ft.

35. $F(x, y, \lambda) = f(x, y) + \lambda(c - ax - by)$

The values of x and y that minimize production subject to the cost constraint satisfy

$$\left.\begin{aligned} \frac{\partial F}{\partial x}(x, y) &= \frac{\partial f}{\partial x}(x, y) - \lambda a = 0 \\ \frac{\partial F}{\partial y}(x, y) &= \frac{\partial f}{\partial y}(x, y) - \lambda b = 0 \\ \frac{\partial F}{\partial \lambda}(x, y) &= c - ax - by = 0 \end{aligned}\right\} \left.\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \lambda a \\ \frac{\partial f}{\partial y}(x, y) &= \lambda b \end{aligned}\right\} \text{Dividing the 1st equation by the second gives } \frac{\frac{\partial f}{\partial x}(x, y)}{\frac{\partial f}{\partial y}(x, y)} = \frac{a}{b}$$

36. Given $f(x, y) = kx^{\alpha}y^{\beta}$, $\frac{\partial f}{\partial x} = \alpha kx^{\alpha-1}y^{\beta}$ and $\frac{\partial f}{\partial y} = \beta kx^{\alpha}y^{\beta-1}$.

Applying the result of Exercise 35, at the optimal values (x, y), $\frac{\alpha kx^{\alpha-1}y^{\beta}}{\beta kx^{\alpha}y^{\beta-1}} = \frac{a}{b}$; so $\frac{\alpha y}{\beta x} = \frac{a}{b}$ or $\frac{y}{x} = \frac{\beta a}{\alpha b}$.