# Homework 30 Solutions Math 141

**1.** $F(x, y, \lambda) = x^2 + 3y^2 + 10 + \lambda(8 - x - y)$

$$\left.\begin{array}{l}\frac{\partial F}{\partial x} = 2x - \lambda = 0 \\ \frac{\partial F}{\partial y} = 6y - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 8 - x - y = 0\end{array}\right\} \quad \left.\begin{array}{l}\lambda = 2x \\ \lambda = 6y \\ 8 - x - y = 0\end{array}\right\} \quad \left.\begin{array}{l}2x = 6y \Rightarrow x = 3y \\ 8 - 3y - y = 0\end{array}\right\} \quad \begin{array}{l}x = 6 \\ y = 2 \\ \lambda = 12\end{array}$$

The minimum value is $6^2 + 3 \cdot 2^2 + 10 = 58.$

**2.** $F(x, y, \lambda) = x^2 - y^2 + \lambda(2x + y - 3)$

$$\left.\begin{array}{l}\frac{\partial F}{\partial x} = 2x + 2\lambda = 0 \\ \frac{\partial F}{\partial y} = -2y + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 2x + y - 3 = 0\end{array}\right\} \quad \left.\begin{array}{l}\lambda = -x \\ \lambda = 2y \\ 2x + y - 3 = 0\end{array}\right\} \quad \left.\begin{array}{l}x = -2y \\ -4y + y - 3 = 0\end{array}\right\} \quad \begin{array}{l}x = 2 \\ y = -1 \\ \lambda = -2\end{array}$$

The maximum value is $2^2 - (-1)^2 = 3.$

**3.** $F(x, y, \lambda) = x^2 + xy - 3y^2 + \lambda(2 - x - 2y)$

$$\left.\begin{array}{l}\frac{\partial F}{\partial x} = 2x + y - \lambda = 0 \\ \frac{\partial F}{\partial y} = x - 6y - 2\lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 2 - x - 2y = 0\end{array}\right\} \quad \left.\begin{array}{l}\lambda = 2x + y \\ \lambda = \frac{1}{2}x - 3y \\ 2 - x - 2y = 0\end{array}\right\} \quad \left.\begin{array}{l}\frac{3}{2}x + 4y = 0 \\ x + 2y = 2\end{array}\right\} \quad \begin{array}{l}x = 8 \\ y = -3 \\ \lambda = 13\end{array}$$

The maximum value is $8^2 + 8(-3) - 3(-3)^2 = 13$.

**4.** $F(x,\ y,\ \lambda) = \frac{1}{2}x^2 - 3xy + y^2 + \frac{1}{2} + \lambda(3x - y - 1)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = x - 3y + 3\lambda = 0 \\ \frac{\partial F}{\partial y} = -3x + 2y - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 3x - y - 1 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -\frac{1}{3}x + y \\ \lambda = -3x + 2y \\ 3x - y - 1 = 0 \end{array}\right\} \left.\begin{array}{l} \frac{8}{3}x - y = 0 \\ 3x - y = 1 \end{array}\right\} \begin{array}{l} x = 3 \\ y = 8 \\ \lambda = 7 \end{array}$$

The minimum value is $\frac{1}{2}\left(3^2\right) - 3(3)(8) + 8^2 + \frac{1}{2} = -3$.

**5.** $F(x,\ y,\ \lambda) = -2x^2 - 2xy - \frac{3}{2}y^2 + x + 2y + \lambda\left(x + y - \frac{5}{2}\right)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = -4x - 2y + 1 + \lambda = 0 \\ \frac{\partial F}{\partial y} = -2x - 3y + 2 + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = x + y - \frac{5}{2} = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = 4x + 2y - 1 \\ \lambda = 2x + 3y - 2 \\ x + y = \frac{5}{2} \end{array}\right\} \left.\begin{array}{l} 2x - y = -1 \\ x + y = \frac{5}{2} \end{array}\right\} \begin{array}{l} x = \frac{1}{2} \\ y = 2 \end{array}$$

**6.** $F(x,\ y,\ \lambda) = x^2 + xy + y^2 - 2x - 5y + \lambda(1 - x + y)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = 2x + y - 2 - \lambda = 0 \\ \frac{\partial F}{\partial y} = x + 2y - 5 + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 1 - x + y = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = 2x + y - 2 \\ \lambda = -x - 2y + 5 \\ 1 - x + y = 0 \end{array}\right\} \left.\begin{array}{l} 3x + 3y = 7 \\ -x + y = -1 \end{array}\right\} \begin{array}{l} x = \frac{5}{3} \\ y = \frac{2}{3} \end{array}$$

**7.** Minimize $xy + y^2 - x - 1$ subject to the constraint $x - 2y = 0$.

$F(x, y, \lambda) = xy + y^2 - x - 1 + \lambda(x - 2y)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = y - 1 + \lambda = 0 \\ \frac{\partial F}{\partial y} = x + 2y - 2\lambda = 0 \\ \frac{\partial F}{\partial \lambda} = x - 2y = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -y + 1 \\ \lambda = \frac{x + 2y}{2} \\ x = 2y \end{array}\right\} \left.\begin{array}{l} x + 4y = 2 \\ x = 2y \end{array}\right\} \begin{array}{l} y = \frac{1}{3} \\ x = \frac{2}{3} \end{array}$$

**8.** Minimize $x^2 - 2xy + 2y^2$ subject to the constraint $2x - y + 5 = 0$.

$F(x, y, \lambda) = x^2 - 2xy + 2y^2 + \lambda(2x - y + 5)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = 2x - 2y + 2\lambda = 0 \\ \frac{\partial F}{\partial y} = -2x + 4y - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 2x - y + 5 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -x + y \\ \lambda = -2x + 4y \\ 2x - y = -5 \end{array}\right\} \left.\begin{array}{l} x = 3y \\ 2x - y = -5 \end{array}\right\} \begin{array}{l} x = -3 \\ y = -1 \end{array}$$

9. Minimize $2x^2 + xy + y^2 - y$ subject to the constraint $x + y = 0$.

$$F(x, y, \lambda) = 2x^2 + xy + y^2 - y + \lambda(x + y)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = 4x + y + \lambda = 0 \\ \dfrac{\partial F}{\partial y} = x + 2y - 1 + \lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x + y = 0 \end{array}\right\} \left.\begin{array}{c} \lambda = -4x - y \\ \lambda = -x - 2y + 1 \\ x = -y \end{array}\right\} \left.\begin{array}{c} -3x = -y + 1 \\ x = -y \end{array}\right\} \begin{array}{l} x = -\dfrac{1}{4} \\ y = \dfrac{1}{4} \end{array}$$

10. Minimize $2x^2 - 2xy + y^2 - 2x + 1$ subject to the constraint $x - y = 3$.

$$F(x, y, \lambda) = 2x^2 - 2xy + y^2 - 2x + 1 + \lambda(x - y - 3)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = 4x - 2y - 2 + \lambda = 0 \\ \dfrac{\partial F}{\partial y} = -2x + 2y - \lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x - y - 3 = 0 \end{array}\right\} \left.\begin{array}{c} \lambda = -4x + 2y + 2 \\ \lambda = -2x + 2y \\ x = y + 3 \end{array}\right\} \begin{array}{l} x = 1 \\ y = -2 \end{array}$$

11. Minimize $18x^2 + 12xy + 4y^2 + 6x - 4y + 5$ subject to the constraint $3x + 2y - 1 = 0$.

$$F(x, y, \lambda) = 18x^2 + 12xy + 4y^2 + 6x - 4y + 5 + \lambda(3x + 2y - 1)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = 36x + 12y + 6 + 3\lambda = 0 \\ \dfrac{\partial F}{\partial y} = 12x + 8y - 4 + 2\lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = 3x + 2y - 1 = 0 \end{array}\right\} \left.\begin{array}{c} \lambda = -12x - 4y - 2 \\ \lambda = -6x - 4y + 2 \\ y = \dfrac{-3x + 1}{2} \end{array}\right\} \left.\begin{array}{c} x = -\dfrac{2}{3} \\ y = \dfrac{-3x + 1}{2} \end{array}\right\} \begin{array}{l} x = -\dfrac{2}{3} \\ y = \dfrac{3}{2} \end{array}$$

12. Minimize $3x^2 - 2xy + x - 3y + 1$ subject to the constraint $x - 3y = 1$.

$$F(x, y, \lambda) = 3x^2 - 2xy + x - 3y + 1 + \lambda(x - 3y - 1)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = 6x - 2y + 1 + \lambda = 0 \\ \dfrac{\partial F}{\partial y} = -2x - 3 - 3\lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x - 3y - 1 = 0 \end{array}\right\} \left.\begin{array}{c} \lambda = -6x + 2y - 1 \\ \lambda = \dfrac{-2x - 3}{3} \\ y = \dfrac{x - 1}{3} \end{array}\right\} \left.\begin{array}{c} y = \dfrac{8x}{3} \\ y = \dfrac{x - 1}{3} \end{array}\right\} \begin{array}{l} x = -\dfrac{1}{7} \\ y = -\dfrac{8}{21} \end{array}$$

13. Minimize $x - xy + 2y^2$ subject to the constraint $x - y + 1 = 0$.

$$F(x, y, \lambda) = x - xy + 2y^2 + \lambda(x - y + 1)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = 1 - y + \lambda = 0 \\ \dfrac{\partial F}{\partial y} = -x + 4y - \lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x - y + 1 = 0 \end{array}\right\} \left.\begin{array}{c} \lambda = y - 1 \\ \lambda = -x + 4y \\ y = x + 1 \end{array}\right\} \left.\begin{array}{c} x = 3y + 1 \\ y = x + 1 \end{array}\right\} \begin{array}{l} x = -2 \\ y = -1 \end{array}$$

**14.** Maximize $xy$ subject to the constraint $x^2 - y = 3$.

$F(x, y, \lambda) = xy + \lambda(x^2 - y - 3)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= y + 2\lambda x = 0 \\ \frac{\partial F}{\partial y} &= x - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= x^2 - y - 3 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= -\frac{y}{2x} \\ \lambda &= x \\ y &= x^2 - 3 \end{aligned}\right\} \left.\begin{aligned} y &= -2x^2 \\ y &= x^2 - 3 \end{aligned}\right\} \begin{aligned} &x = -1 \text{ or } && x = 1 \\ &y = -2 && y = -2 \end{aligned}$$

Now check both answers in the original function to see which pair maximizes $xy$. The pair that maximizes $xy$ is $x = -1, y = -2$.

**15.** Minimize $xy + xz - yz$ subject to the constraint $x + y + z = 1$.

$F(x, y, z, \lambda) = xy + xz - yz + \lambda(x + y + z - 1)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= y + z + \lambda = 0 \\ \frac{\partial F}{\partial y} &= x - z + \lambda = 0 \\ \frac{\partial F}{\partial z} &= x - y + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= x + y + z - 1 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= -y - z \\ \lambda &= -x + z \\ \lambda &= -x + y \\ x + y + z &= 1 \end{aligned}\right\} \left.\begin{aligned} x - y &= 2z \\ y &= z \\ x + y + z &= 1 \end{aligned}\right\} \left.\begin{aligned} x = y + 2z &\Rightarrow x = 3y \\ y &= z \\ 3y + y + y &= 1 \end{aligned}\right\} \begin{aligned} x &= \frac{3}{5} \\ y &= \frac{1}{5} \\ z &= \frac{1}{5} \end{aligned}$$

**16.** Minimize $xy + xz - 2yz$ subject to the constraint $x + y + z = 2$.

$F(x, y, z, \lambda) = xy + xz - 2yz + \lambda(x + y + z - 2)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= y + z + \lambda = 0 \\ \frac{\partial F}{\partial y} &= x - 2z + \lambda = 0 \\ \frac{\partial F}{\partial z} &= x - 2y + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= x + y + z - 2 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= -y - z \\ \lambda &= -x + 2z \\ \lambda &= -x + 2y \\ x + y + z &= 2 \end{aligned}\right\} \left.\begin{aligned} x - y &= 3z \\ y &= z \\ x + y + z &= 2 \end{aligned}\right\} \left.\begin{aligned} x = y + 3z &\Rightarrow x = 4y \\ y &= z \\ 4y + y + y &= 2 \end{aligned}\right\} \begin{aligned} x &= \frac{4}{3} \\ y &= \frac{1}{3} \\ z &= \frac{1}{3} \end{aligned}$$

**17.** We want to minimize the function $x + y$ subject to the constraint $xy = 25$ or $xy - 25 = 0$.

$F(x, y, \lambda) = x + y + \lambda(xy - 25)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 1 + \lambda y = 0 \\ \frac{\partial F}{\partial y} &= 1 + \lambda x = 0 \\ \frac{\partial F}{\partial \lambda} &= xy - 25 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= \frac{-1}{y} \\ \lambda &= \frac{-1}{x} \\ xy - 25 &= 0 \end{aligned}\right\} \left.\begin{aligned} x - y &= 0 \\ xy &= 25 \end{aligned}\right\} x^2 = 25 \text{ or } x = \pm 5$$

so $x = 5, y = 5$ (the positive numbers).