Homework 29 Solutions Math 141

29. $f(x, y) = x^4 - x^2 - 2xy + y^2 + 1$;

$\frac{\partial f}{\partial x} = 4x^3 - 2x - 2y$; $\frac{\partial f}{\partial y} = 2y - 2x$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 2$; $\frac{\partial^2 f}{\partial y^2} = 2$; $\frac{\partial^2 f}{\partial x \partial y} = -2$

Solving the system

$$\left.\begin{aligned} 4x^3 - 2x - 2y &= 0 \\ -2x + 2y &= 0 \end{aligned}\right\}$$

yields the solutions $(0, 0)$, $(-1, -1)$, and $(1, 1)$.

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$= \left(12x^2 - 2\right)2 - (-2)^2$$
$$= 24x^2 - 8$$

$D(0, 0) = -8 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(0, 0)$.

$D(-1, -1) = 16 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 10 > 0$ so $f(x, y)$ has a relative minimum at $(-1, -1)$.

$D(1, 1) = 16 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 10 > 0$ so $f(x, y)$ has a relative minimum at $(1, 1)$.

30. $f(x, y) = x^2 + 2xy + 10y^2$;

$\frac{\partial f}{\partial x} = 2x + 2y$; $\frac{\partial f}{\partial y} = 20y + 2x$

$\frac{\partial^2 f}{\partial x^2} = 2$; $\frac{\partial^2 f}{\partial y^2} = 20$; $\frac{\partial^2 f}{\partial x \partial y} = 2$

Solving the system

$$\left.\begin{aligned} 2x + 2y &= 0 \\ 2x + 20y &= 0 \end{aligned}\right\}$$

yields the solution $(0, 0)$.

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$= 2(20) - 2^2 = 36$$

$D(0, 0) = 38 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 2 > 0$, so $f(x, y)$ has a relative minimum at $(0, 0)$.

31. $f(x, y) = 6xy - 3y^2 - 2x + 4y - 1$;

$\frac{\partial f}{\partial x} = 6y - 2$; $\frac{\partial f}{\partial y} = -6y + 6x + 4$

$\frac{\partial^2 f}{\partial x^2} = 0$; $\frac{\partial^2 f}{\partial y^2} = -6$; $\frac{\partial^2 f}{\partial x \partial y} = 6$

Solving the system

$$\left.\begin{aligned} 6y - 2 &= 0 \\ 6x - 6y + 4 &= 0 \end{aligned}\right\}$$

yields the solution $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$= 0(-6) - 6^2 = -36$$

$D\left(-\frac{1}{3}, \frac{1}{3}\right) = -36 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

32. $f(x, y) = 2xy + y^2 + 2x - 1$

$\frac{\partial f}{\partial x} = 2y + 2$; $\frac{\partial f}{\partial y} = 2x + 2y$

$\frac{\partial^2 f}{\partial x^2} = 0$; $\frac{\partial^2 f}{\partial y^2} = 2$; $\frac{\partial^2 f}{\partial x \partial y} = 2$

Solving the system

$$\left.\begin{aligned} 2y + 2 &= 0 \\ 2x + 2y &= 0 \end{aligned}\right\}$$

yields the solution $(1, -1)$.

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$= 0(2) - 2^2 = -4$$

$D(1, -1) = -4 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(1, -1)$.

33. $f(x, y) = -2x^2 + 2xy - 25y^2 - 2x + 8y - 1$

$\frac{\partial f}{\partial x} = -4x + 2y - 2$; $\frac{\partial f}{\partial y} = 2x - 50y + 8$

$\frac{\partial^2 f}{\partial x^2} = -4$; $\frac{\partial^2 f}{\partial y^2} = -50$; $\frac{\partial^2 f}{\partial x \partial y} = 2$

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Solving the system

$$\begin{rcases} -4x+2y-2=0 \\ 2x-50y+8=0 \end{rcases}$$

yields the solution $\left(-\frac{3}{7}, \frac{1}{7}\right)$.

$$D(x,y)=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$=-4(-50)-2^2=196$$

$D\left(-\frac{3}{7}, \frac{1}{7}\right)=196>0$ and $\frac{\partial^2 f}{\partial x^2}\left(-\frac{3}{7}, \frac{1}{7}\right)=-4<0$, so $f(x, y)$ has a relative maximum at $\left(-\frac{3}{7}, \frac{1}{7}\right)$.

34. $f(x, y)=3x^2+8xy-3y^2-2x+4y+1$

$\frac{\partial f}{\partial x}=6x+8y-2;\ \frac{\partial f}{\partial y}=8x-6y+4$

$\frac{\partial^2 f}{\partial x^2}=6;\ \frac{\partial^2 f}{\partial y^2}=-6;\ \frac{\partial^2 f}{\partial x \partial y}=8$

Solving the system

$$\begin{rcases} 6x+8y-2=0 \\ 8x-6y+4=0 \end{rcases}$$

yields the solution $\left(-\frac{1}{5}, \frac{2}{5}\right)$.

$$D(x,y)=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$=6(-6)-8^2=-100$$

$D\left(-\frac{1}{5}, \frac{2}{5}\right)=-100<0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $\left(-\frac{1}{5}, \frac{2}{5}\right)$.

35. $f(x, y)=x^4-12x^2-4xy-y^2+16$

$\frac{\partial f}{\partial x}=4x^3-24x-4y;\ \frac{\partial f}{\partial y}=-4x-2y$

$\frac{\partial^2 f}{\partial x^2}=12x^2-24;\ \frac{\partial^2 f}{\partial y^2}=-2;\ \frac{\partial^2 f}{\partial x \partial y}=-4$

Solving the system

$$\begin{rcases} 4x^3-24x-4y=0 \\ -4x-2y=0 \end{rcases}$$

yields the solutions $(0, 0)$, $(-2, 4)$, and $(2, -4)$.

$$D(x,y)=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$=\left(12x^2-24\right)(-2)-(-4)^2$$
$$=-24x^2+32$$

$D(0, 0)=32>0$ and $\frac{\partial^2 f}{\partial x^2}(0, 0)=-24<0$, so $f(x, y)$ has a relative maximum at $(0, 0)$.

$D(-2, 4)=-24(-2)^2+32=-64<0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(-2, 4)$.

$D(2, -4)=-24(2)^2+32=-64<0$, and $\frac{\partial^2 f}{\partial x^2}=24>0$ so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(2, -4)$.

36. $f(x, y)=\frac{17}{4}x^2+2xy+5y^2+5x-2y+2$

$\frac{\partial f}{\partial x}=\frac{17}{2}x+2y+5;\ \frac{\partial f}{\partial y}=2x+10y-2$

$\frac{\partial^2 f}{\partial x^2}=\frac{17}{2};\ \frac{\partial^2 f}{\partial y^2}=10;\ \frac{\partial^2 f}{\partial x \partial y}=2$

Solving the system

$$\begin{rcases} \frac{17}{2}x+2y+5=0 \\ 2x+10y-2=0 \end{rcases}$$

yields the solution $\left(-\frac{2}{3}, \frac{1}{3}\right)$.

$$D(x,y)=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$=\frac{17}{2}(10)-2^2=81$$

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$D\left(-\frac{2}{3}, \frac{1}{3}\right) = 81 > 0$ and

$\frac{\partial^2 f}{\partial x^2}\left(-\frac{2}{3}, \frac{1}{3}\right) = \frac{17}{2} > 0$, so $f(x, y)$ has a relative minimum at $\left(-\frac{2}{3}, \frac{1}{3}\right)$.

37. $f(x, y) = x^2 - 2xy + 4y^2$

$\frac{\partial f}{\partial x} = 2x - 2y; \frac{\partial f}{\partial y} = -2x + 8y$

$\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2} = 8; \frac{\partial^2 f}{\partial x \partial y} = -2$

$\left.\begin{aligned} 2x - 2y &= 0 \\ -2x + 8y &= 0 \end{aligned}\right\} \begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2 \cdot 8 - (-2)^2 = 12$$

$D(0, 0) = 2 \cdot 8 - (-2)^2 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0, 0) > 0$, so $f(x, y)$ has a relative minimum at (0, 0).

38. $f(x, y) = 2x^2 + 3xy + 5y^2$

$\frac{\partial f}{\partial x} = 4x + 3y; \frac{\partial f}{\partial y} = 3x + 10y$

$\frac{\partial^2 f}{\partial x^2} = 4; \frac{\partial^2 f}{\partial y^2} = 10; \frac{\partial^2 f}{\partial x \partial y} = 3$

$\left.\begin{aligned} 4x + 3y &= 0 \\ 3x + 10y &= 0 \end{aligned}\right\} \begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= 4 \cdot 10 - 3^2 = 31 \end{aligned}$$

$D(0, 0) = 4 \cdot 10 - 3^2 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0, 0) > 0$, so $f(x, y)$ has a relative minimum at (0, 0).

39. $f(x, y) = -2x^2 + 2xy - y^2 + 4x - 6y + 5$

$\frac{\partial f}{\partial x} = -4x + 2y + 4; \frac{\partial f}{\partial y} = 2x - 2y - 6$

$\frac{\partial^2 f}{\partial x^2} = -4; \frac{\partial^2 f}{\partial y^2} = -2; \frac{\partial^2 f}{\partial x \partial y} = 2$

$\left.\begin{aligned} -4x + 2y + 4 &= 0 \\ 2x - 2y - 6 &= 0 \end{aligned}\right\} \begin{aligned} x &= -1 \\ y &= -4 \end{aligned}$

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= -4(-2) - 2^2 = 4 \end{aligned}$$

$D(-1, -4) = (-4)(-2) - 2^2 > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, -4) < 0$, so $f(x, y)$ has a relative maximum at (–1, –4).

40. $f(x, y) = -x^2 - 8xy - y^2$

$\frac{\partial f}{\partial x} = -2x - 8y; \frac{\partial f}{\partial y} = -8x - 2y$

$\frac{\partial^2 f}{\partial x^2} = -2; \frac{\partial^2 f}{\partial y^2} = -2; \frac{\partial^2 f}{\partial x \partial y} = -8$

$\left.\begin{aligned} -2x - 8y &= 0 \\ -8x - 2y &= 0 \end{aligned}\right\} \begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= -2(-2) - (-8)^2 = -60 \end{aligned}$$

$D(0, 0) = (-2)(-2) - (-8)^2 < 0$, so $f(x, y)$ has neither a maximum nor a minimum at (0, 0).

41. $f(x, y) = x^2 + 2xy + 5y^2 + 2x + 10y - 3$

$\frac{\partial f}{\partial x} = 2x + 2y + 2; \frac{\partial f}{\partial y} = 2x + 10y + 10$

$\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2} = 10; \frac{\partial^2 f}{\partial x \partial y} = 2$

$\left.\begin{aligned} 2x + 2y + 2 &= 0 \\ 2x + 10y + 10 &= 0 \end{aligned}\right\} \begin{aligned} x &= 0 \\ y &= -1 \end{aligned}$

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= 2(10) - 2^2 = 16 \end{aligned}$$

$D(0, -1) = (2)(10) - 2^2 > 0, \frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at (0, –1).

42. $f(x, y) = x^2 - 2xy + 3y^2 + 4x - 16y + 22$

$$\frac{\partial f}{\partial x} = 2x - 2y + 4; \frac{\partial f}{\partial y} = -2x + 6y - 16$$

$$\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2} = 6; \frac{\partial^2 f}{\partial x \partial y} = -2$$

$$\left.\begin{array}{r} 2x - 2y + 4 = 0 \\ -2x + 6y - 16 = 0 \end{array}\right\} \begin{array}{l} x = 1 \\ y = 3 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2(6) - (-2)^2 = 8$$

$D(1, 3) = 2 \cdot 6 - (-2)^2 > 0, \frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at (1, 3).

43. $f(x, y) = x^3 - y^2 - 3x + 4y$

$$\frac{\partial f}{\partial x} = 3x^2 - 3; \frac{\partial f}{\partial y} = -2y + 4$$

$$\frac{\partial^2 f}{\partial x^2} = 6x; \frac{\partial^2 f}{\partial y^2} = -2; \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\left.\begin{array}{r} 3x^2 - 3 = 0 \\ -2y + 4 = 0 \end{array}\right\} \begin{array}{l} x = \pm 1 \\ y = 2 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 6x(-2) - 0^2 = -12x$$

$D(1, 2) < 0$, so $f(x, y)$ has neither a maximum nor a minimum at (1, 2).

$D(-1, 2) > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, 2) = -6 < 0$, so $f(x, y)$ has a relative maximum at (–1, 2).

44. $f(x, y) = x^3 - 2xy + 4y$

$$\frac{\partial f}{\partial x} = 3x^2 - 2y; \frac{\partial f}{\partial y} = -2x + 4$$

$$\frac{\partial^2 f}{\partial x^2} = 6x; \frac{\partial^2 f}{\partial y^2} = 0; \frac{\partial^2 f}{\partial x \partial y} = -2$$

$$\left.\begin{array}{r} 3x^2 - 2y = 0 \\ -2x + 4 = 0 \end{array}\right\} \begin{array}{l} x = 2 \\ y = 6 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 6x(0) - (-2)^2 = -4$$

$D(2, 6) < 0$, so $f(x, y)$ has neither a maximum nor a minimum at (2, 6).

45. $f(x, y) = 2x^2 + y^3 - x - 12y + 7$

$$\frac{\partial f}{\partial x} = 4x - 1; \frac{\partial f}{\partial y} = 3y^2 - 12$$

$$\frac{\partial^2 f}{\partial x^2} = 4; \frac{\partial^2 f}{\partial y^2} = 6y; \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\left.\begin{array}{r} 4x - 1 = 0 \\ 3y^2 - 12 = 0 \end{array}\right\} \begin{array}{l} x = 1/4 \\ y = \pm 2 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4(6y) - 0^2 = 24y$$

$D\left(\frac{1}{4}, 2\right) = 48 > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at $\left(\frac{1}{4}, 2\right)$.

$D\left(\frac{1}{4}, -2\right) = -48 < 0$, so $f(x, y)$ has neither a maximum nor a minimum at $\left(\frac{1}{4}, -2\right)$.

46. $f(x, y) = x^2 + 4xy + 2y^4$

$$\frac{\partial f}{\partial x} = 2x + 4y; \frac{\partial f}{\partial y} = 4x + 8y^3$$

$$\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2} = 24y^2; \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$\left.\begin{array}{r} 2x + 4y = 0 \\ 4x + 8y^3 = 0 \end{array}\right\} \begin{array}{l} x = -2y \\ 8y^3 - 8y = 0 \end{array} \Rightarrow$$

$$8y\left(y^2 - 1\right) = 0 \Rightarrow y = 0, \pm 1$$

Solutions: (0, 0), (–2, 1), (2, –1)

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2\left(24y^2\right) - 4^2 = 48y^2 - 16$$

$D(0, 0) < 0$, so $f(x, y)$ has neither a relative minimum nor a relative maximum at (0, 0).

$D(-2, 1) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at (–2, 1).

$D(2, -1) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at (2, –1).

47. $f(x, y, z) = 2x^2 + 3y^2 + z^2 - 2x - y - z$

$$\frac{\partial f}{\partial x} = 4x - 2; \frac{\partial f}{\partial y} = 6y - 1; \frac{\partial f}{\partial z} = 2z - 1$$

$$\left.\begin{aligned} 4x - 2 &= 0 \\ 6y - 1 &= 0 \\ 2z - 1 &= 0 \end{aligned}\right\} x = \frac{1}{2}; y = \frac{1}{6}; z = \frac{1}{2}$$

$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right)$ is the only point at which $f(x, y, z)$ can have a relative minimum.

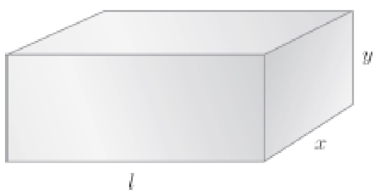
48. $f(x, y, z) = 5 + 8x - 4y + x^2 + y^2 + z^2$

$$\frac{\partial f}{\partial x} = 8 + 2x; \quad \frac{\partial f}{\partial y} = 2y - 4; \quad \frac{\partial f}{\partial z} = 2z$$

$$\left.\begin{aligned} 8 + 2x &= 0 \\ 2y - 4 &= 0 \\ 2z &= 0 \end{aligned}\right\} \begin{aligned} x &= -4 \\ y &= 2 \\ z &= 0 \end{aligned}$$

$(-4, 2, 0)$ is the only point at which $f(x, y, z)$ can have a relative minimum.

49.



Let x, y, and l be as shown in the figure.
Since $l = 84 - 2x - 2y$, the volume of the box may be written as

$$V(x, y) = xy(84 - 2x - 2y) = 84xy - 2x^2y - 2xy^2.$$

$$\frac{\partial V}{\partial x} = 84y - 4xy - 2y^2; \frac{\partial V}{\partial y} = 84x - 2x^2 - 4xy$$

$$\frac{\partial^2 V}{\partial x^2} = -4y; \frac{\partial^2 V}{\partial y^2} = -4x; \frac{\partial^2 V}{\partial x \partial y} = 84 - 4x - 4y$$

$$\left.\begin{aligned} 84y - 4xy - 2y^2 &= 0 \\ 84x - 2x^2 - 4xy &= 0 \end{aligned}\right\} \begin{aligned} x &= 0 \\ y &= 0 \end{aligned} \quad \text{or} \quad \left.\begin{aligned} 84 - 4x - 2y &= 0 \\ 84 - 4y - 2x &= 0 \end{aligned}\right\} \begin{aligned} x &= 14 \\ y &= 14 \end{aligned}$$

Obviously, $(0, 0)$ does not give the maximum value of $V(x, y)$. To verify that $(14, 14)$ is the maximum, check

$$D = \frac{\partial^2 V}{\partial x^2} \cdot \frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 = -4y(-4x) - (84 - 4x - 4y)^2$$

$$D(14, 14) = -4(14)(-4)(14) - [84 - 4(14) - 4(14)]^2 > 0 \text{ and } \frac{\partial^2 V}{\partial x^2} = -4(14) < 0.$$

Thus, the dimensions that give the maximum volume are $x = 14$, $y = 14$, $l = 84 - 56 = 28$; or $14 \times 14 \times 28$ in.

50. Let x, y, and z be the dimensions of the box. Since the volume of the box is 1000 in^3, $x > 0$, $y > 0$, and $z = \frac{1000}{xy}$. The surface area is $S(x, y) = 2xy + 2x\left(\frac{1000}{xy}\right) + 2y\left(\frac{1000}{xy}\right) = 2xy + \frac{2000}{y} + \frac{2000}{x}$.

$$\frac{\partial S}{\partial x} = 2y - \frac{2000}{x^2}; \frac{\partial S}{\partial y} = 2x - \frac{2000}{y^2}; \frac{\partial^2 S}{\partial x^2} = \frac{4000}{x^3}; \frac{\partial^2 S}{\partial y^2} = \frac{4000}{y^3}; \frac{\partial^2 S}{\partial x \partial y} = 2$$

$$\left.\begin{aligned} 2y - \frac{2000}{x^2} = 0 \\ 2x - \frac{2000}{y^2} = 0 \end{aligned}\right\} x = y = 10$$

To verify that (10, 10) is a minimum, check

$$D = \frac{\partial^2 S}{\partial x^2} \cdot \frac{\partial^2 S}{\partial y^2} - \left(\frac{\partial^2 S}{\partial x \partial y}\right)^2 = \frac{4000}{x^3} \cdot \frac{4000}{y^3} - (2)^2 = \frac{4000^2}{x^3 y^3} - 4$$

$$D(10, 10) = \frac{4000}{1000^2}^2 - 2^2 > 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(10, 10) > 0.$$

Thus, the dimensions giving the smallest surface area are $10 \times 10 \times 10$ in.

51. The revenue is $10x + 9y$, so the profit function is

$$\begin{aligned} P(x, y) &= 10x + 9y - \left[400 + 2x + 3y + 0.01(3x^2 + xy + 3y^2)\right] \\ &= 8x + 6y - 0.03x^2 - 0.01xy - 0.03y^2 - 400 \end{aligned}$$

$$\frac{\partial P}{\partial x} = 8 - 0.06x - 0.01y; \frac{\partial P}{\partial y} = 6 - 0.01x - 0.06y; \quad \frac{\partial^2 P}{\partial x^2} = -0.06; \frac{\partial^2 P}{\partial y^2} = -0.06; \frac{\partial^2 P}{\partial x \partial y} = -0.01$$

$$\left.\begin{aligned} 8 - 0.06x - 0.01y = 0 \\ 6 - 0.01x - 0.06y = 0 \end{aligned}\right\} \begin{aligned} x = 120 \\ y = 80 \end{aligned} \Rightarrow (120, 80) \text{ is a maximum. Verify this by checking}$$

$$D = \frac{\partial^2 P}{\partial x^2} \cdot \frac{\partial^2 P}{\partial y^2} - \left(\frac{\partial^2 P}{\partial x \partial y}\right)^2 = -0.06(-0.06) - (-0.01)^2 = 0.0035 \Rightarrow D(120, 80) > 0.$$

$$\frac{\partial^2 P}{\partial x^2}(120, 80) = -0.06 < 0.$$

Thus profit is maximized by producing 120 units of product I and 80 units of product II.

52. The cost is $30x + 20y$, so the profit function is

$$P(x, y) = 98x + 112y - 0.04xy - 0.1x^2 - 0.2y^2 - 30x - 20y = 68x + 92y - 0.04xy - 0.1x^2 - 0.2y^2$$

$$\frac{\partial P}{\partial x} = 68 - 0.04y - 0.2x; \frac{\partial P}{\partial y} = 92 - 0.04x - 0.4y; \quad \frac{\partial^2 P}{\partial x^2} = -0.2; \frac{\partial^2 P}{\partial y^2} = -0.4; \frac{\partial^2 P}{\partial x \partial y} = -0.04$$

$$\left.\begin{aligned} 68 - 0.04y - 0.2x = 0 \\ 92 - 0.04x - 0.4y = 0 \end{aligned}\right\} \begin{aligned} x = 300 \\ y = 200 \end{aligned}$$

To verify that (300, 200) is a maximum, check

$$D = \frac{\partial^2 P}{\partial x^2} \cdot \frac{\partial^2 P}{\partial y^2} - \left(\frac{\partial^2 P}{\partial x \partial y}\right)^2 = -0.2(-0.4) - (-0.04)^2 = 0.0784 \Rightarrow D(300, 200) > 0.$$

$$\frac{\partial^2 P}{\partial x^2} < 0.$$

Thus the profit is maximized by producing 300 units of product I and 200 units of product II.