Homework 28 Solutions Math 141

1. $f(x, y) = x^2 - 3y^2 + 4x + 6y + 8$

$\frac{\partial f}{\partial x} = 2x + 4; \frac{\partial f}{\partial y} = -6y + 6$

$\left.\begin{array}{r} 2x + 4 = 0 \\ -6y + 6 = 0 \end{array}\right\} \begin{array}{l} x = -2 \\ y = 1 \end{array}$

The only possible extreme point is $(-2, 1)$.

2. $f(x, y) = \frac{1}{2}x^2 + y^2 - 3x + 2y - 5$

$\frac{\partial f}{\partial x} = x - 3; \frac{\partial f}{\partial y} = 2y + 2$

$\left.\begin{array}{r} x - 3 = 0 \\ 2y + 2 = 0 \end{array}\right\} \begin{array}{l} x = 3 \\ y = -1 \end{array}$

The only possible extreme point is $(3, -1)$.

3. $f(x, y) = x^2 - 5xy + 6y^2 + 3x - 2y + 4$

$\frac{\partial f}{\partial x} = 2x - 5y + 3; \frac{\partial f}{\partial y} = -5x + 12y - 2$

$\left.\begin{array}{r} 2x - 5y + 3 = 0 \\ -5x + 12y - 2 = 0 \end{array}\right\} \begin{array}{l} x = 26 \\ y = 11 \end{array}$

The only possible extreme point is $(26, 11)$.

4. $f(x, y) = -3x^2 + 7xy - 4y^2 + x + y$

$\frac{\partial f}{\partial x} = -6x + 7y + 1; \frac{\partial f}{\partial y} = 7x - 8y + 1$

$\left.\begin{array}{r} -6x + 7y + 1 = 0 \\ 7x - 8y + 1 = 0 \end{array}\right\} \begin{array}{l} x = -15 \\ y = -13 \end{array}$

The only possible extreme point is $(-15, -13)$.

5. $f(x, y) = 3x^2 + 8xy - 3y^2 - 2x + 4y - 1$

$\frac{\partial f}{\partial x} = 6x + 8y - 2; \quad \frac{\partial f}{\partial y} = 8x - 6y + 4$

$\left.\begin{array}{r} 6x + 8y - 2 = 0 \\ 8x - 6y + 4 = 0 \end{array}\right\} \begin{array}{l} x = -\frac{1}{5} \\ y = \frac{2}{5} \end{array}$

The only possible extreme point is $\left(-\frac{1}{5}, \frac{2}{5}\right)$.

6. $f(x, y) = 4x^2 + 4xy - 3y^2 + 4y - 1$

$\frac{\partial f}{\partial x} = 8x + 4y; \quad \frac{\partial f}{\partial y} = 4x - 6y + 4$

$\left.\begin{array}{r} 8x + 4y = 0 \\ 4x - 6y + 4 = 0 \end{array}\right\} \begin{array}{l} x = -\frac{1}{4} \\ y = \frac{1}{2} \end{array}$

The only possible extreme point is $\left(-\frac{1}{4}, \frac{1}{2}\right)$.

7. $f(x, y) = x^3 + y^2 - 3x + 6y$

$\frac{\partial f}{\partial x} = 3x^2 - 3; \frac{\partial f}{\partial y} = 2y + 6$

$\left.\begin{array}{r} 3x^2 - 3 = 0 \\ 2y + 6 = 0 \end{array}\right\} \begin{array}{l} x = \pm 1 \\ y = -3 \end{array}$

The possible extreme points are $(1, -3)$ and $(-1, -3)$.

8. $f(x, y) = x^2 - y^3 + 5x + 12y + 1$

$\frac{\partial f}{\partial x} = 2x + 5; \frac{\partial f}{\partial y} = -3y^2 + 12$

$\left.\begin{array}{r} 2x + 5 = 0 \\ -3y^2 + 12 = 0 \end{array}\right\} \begin{array}{l} x = -\frac{5}{2} \\ y = \pm 2 \end{array}$

The possible extreme points are $\left(-\frac{5}{2}, 2\right)$ and $\left(-\frac{5}{2}, -2\right)$.

9. $f(x, y) = -8y^3 + 4xy + 9y^2 - 2y$

$\frac{\partial f}{\partial x} = 4y; \quad \frac{\partial f}{\partial y} = -24y^2 + 4x + 18y - 2$

$\left.\begin{array}{r} 4y = 0 \\ -24y^2 + 18y + 4x - 2 = 0 \end{array}\right\} \begin{array}{l} x = \frac{1}{2} \\ y = 0 \end{array}$

The only possible extreme point is $\left(\frac{1}{2}, 0\right)$.

10. $f(x, y) = -8y^3 + 4xy + 4x^2 + 9y^2$

$\frac{\partial f}{\partial x} = 8x + 4y; \quad \frac{\partial f}{\partial y} = -24y^2 + 18y + 4x$

$\left.\begin{array}{r} 8x + 4y = 0 \\ -24y^2 + 18y + 4x = 0 \end{array}\right\} \left.\begin{array}{r} x = -\frac{1}{2}y \\ -24y^2 + 18y = -4x \end{array}\right\}$

$-24y^2 + 18y = 2y \Rightarrow y = 0, \frac{2}{3}$

The possible extreme points are $(0, 0)$ and $\left(-\frac{1}{3}, \frac{2}{3}\right)$.

11. $f(x, y) = 2x^3 + 2x^2y - y^2 + y$

$\frac{\partial f}{\partial x} = 6x^2 + 4xy; \quad \frac{\partial f}{\partial y} = 2x^2 - 2y + 1$

$\left.\begin{array}{r} 6x^2 + 4xy = 0 \\ 2x^2 - 2y + 1 = 0 \end{array}\right\} \left.\begin{array}{r} 3x^2 + 2xy = 0 \\ y = x^2 + \frac{1}{2} \end{array}\right\}$

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$$3x^2 + 2x\left(x^2 + \frac{1}{2}\right) = 0 \Rightarrow 2x^3 + 3x^2 + x = 0 \Rightarrow$$
$$x\left(2x^2 + 3x + 1\right) = 0 \Rightarrow x(2x+1)(x+1) = 0 \Rightarrow$$
$$x = 0, -\frac{1}{2}, -1 \Rightarrow y = \frac{1}{2}, \frac{3}{4}, \frac{3}{2}$$

The possible extreme points are $\left(0, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, \frac{3}{4}\right)$, and $\left(-1, \frac{3}{2}\right)$.

12. $f(x, y) = \frac{15}{4}x^2 + 6xy - 3y^2 + 3x + 6y$

$$\frac{\partial f}{\partial x} = \frac{15}{2}x + 6y + 3; \ \frac{\partial f}{\partial y} = 6x - 6y + 6$$
$$\left.\begin{aligned} \frac{15}{2}x + 6y + 3 = 0 \\ 6x - 6y + 6 = 0 \end{aligned}\right\} \left.\begin{aligned} \frac{15}{2}x + 6y + 3 = 0 \\ y = x + 1 \end{aligned}\right\}$$
$$\frac{15}{2}x + 6(x+1) + 3 = 0 \Rightarrow$$
$$\frac{27}{2}x = -9 \Rightarrow x = -\frac{2}{3} \Rightarrow y = -\frac{2}{3} + 1 = \frac{1}{3}$$

The only possible extreme point is $\left(-\frac{2}{3}, \frac{1}{3}\right)$.

13. $f(x, y) = \frac{1}{3}x^3 - 2y^3 - 5x + 6y - 5$

$$\frac{\partial f}{\partial x} = x^2 - 5; \frac{\partial f}{\partial y} = -6y^2 + 6$$
$$\left.\begin{aligned} x^2 - 5 = 0 \\ -6y^2 + 6 = 0 \end{aligned}\right\} \begin{aligned} x = \pm\sqrt{5} \\ y = \pm 1 \end{aligned}$$

There are four possible extreme points: $\left(\sqrt{5}, 1\right), \left(\sqrt{5}, -1\right), \left(-\sqrt{5}, 1\right)$, and $\left(-\sqrt{5}, -1\right)$.

14. $f(x, y) = x^4 - 8xy + 2y^2 - 3$

$$\frac{\partial f}{\partial x} = 4x^3 - 8y; \frac{\partial f}{\partial y} = -8x + 4y$$
$$\left.\begin{aligned} 4x^3 - 8y = 0 \\ -8x + 4y = 0 \end{aligned}\right\} \left.\begin{aligned} y = \frac{1}{2}x^3 \\ y = 2x \end{aligned}\right\} \Rightarrow$$
$$2x = \frac{1}{2}x^3 \Rightarrow x^3 - 4x = 0 \Rightarrow$$
$$x(x-2)(x+2) = 0 \Rightarrow x = 0, \ \pm 2 \Rightarrow$$
$$y = 0, 4, -4$$

The possible extreme points are (0, 0), (2, 4), and (–2, –4).

15. $f(x, y) = x^3 + x^2y - y$

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy; \ \frac{\partial f}{\partial y} = x^2 - 1$$
$$\left.\begin{aligned} 3x^2 + 2xy = 0 \\ x^2 - 1 = 0 \end{aligned}\right\} \left.\begin{aligned} y = \frac{-3x^2}{2x} = -\frac{3}{2}x \\ x^2 = 1 \end{aligned}\right\} \Rightarrow$$
$$x = \pm 1 \Rightarrow y = \mp\frac{3}{2}$$

The possible extreme points are $\left(-1, \frac{3}{2}\right)$ and $\left(1, -\frac{3}{2}\right)$.

16. $f(x, y) = x^4 - 2xy - 7x^2 + y^2 + 3$

$$\frac{\partial f}{\partial x} = 4x^3 - 2y - 14x; \ \frac{\partial f}{\partial y} = -2x + 2y$$
$$\left.\begin{aligned} 4x^3 - 2y - 14x = 0 \\ -2x + 2y = 0 \end{aligned}\right\} \left.\begin{aligned} y = 2x^3 - 7x \\ y = x \end{aligned}\right\} \Rightarrow$$
$$x = 2x^3 - 7x \Rightarrow 2x^3 - 8x = 0 \Rightarrow$$
$$2x\left(x^2 - 4\right) = 0 \Rightarrow x = 0, \pm 2 \Rightarrow y = 0, \pm 2$$

The possible extreme points are (–2, –2), (0, 0), and (2, 2).

17. $f(x, y) = 2x + 3y + 9 - x^2 - xy - y^2$

$$\frac{\partial f}{\partial x} = 2 - 2x - y; \frac{\partial f}{\partial y} = 3 - x - 2y$$
$$\left.\begin{aligned} 2 - 2x - y = 0 \\ 3 - x - 2y = 0 \end{aligned}\right\} \left.\begin{aligned} y = 2 - 2x \\ x = \frac{1}{3} \end{aligned}\right\} \begin{aligned} x = \frac{1}{3} \\ y = \frac{4}{3} \end{aligned}$$

$\left(\frac{1}{3}, \frac{4}{3}\right)$ is the only point at which $f(x, y)$ can have a maximum, so the maximum value must occur at this point.

18. $f(x, y) = \frac{1}{2}x^2 + 2xy + 3y^2 - x + 2y$

$$\frac{\partial f}{\partial x} = x + 2y - 1; \frac{\partial f}{\partial y} = 2x + 6y + 2$$
$$\left.\begin{aligned} x + 2y - 1 = 0 \\ 2x + 6y + 2 = 0 \end{aligned}\right\} \left.\begin{aligned} x = 1 - 2y \\ y = -2 \end{aligned}\right\} \begin{aligned} x = 5 \\ y = -2 \end{aligned}$$

Thus (5, –2) is the only point at which $f(x, y)$ can have a minimum, so the minimum value must occur at this point.

19. $f(x, y) = 3x^2 - 6xy + y^3 - 9y$

$$\frac{\partial f}{\partial x} = 6x - 6y; \frac{\partial f}{\partial y} = -6x + 3y^2 - 9$$

$$\frac{\partial^2 f}{\partial x^2} = 6; \frac{\partial^2 f}{\partial y^2} = 6y; \frac{\partial^2 f}{\partial x \partial y} = -6$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 6 \cdot 6y - (-6)^2 = 36y - 36$$

$D(3, 3) = 36 \cdot 3 - 36 > 0$, and $\frac{\partial^2 f}{\partial x^2}(3, 3) > 0$ so $(3, 3)$ is a relative minimum of $f(x, y)$. $D(-1, -1) = 36(-1) - 36 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(-1, -1)$.

20. $f(x, y) = 6xy^2 - 2x^3 - 3y^4$

$$\frac{\partial f}{\partial x} = 6y^2 - 6x^2; \frac{\partial f}{\partial y} = 12xy - 12y^3$$

$$\frac{\partial^2 f}{\partial x^2} = -12x; \frac{\partial^2 f}{\partial y^2} = 12x - 36y^2; \frac{\partial^2 f}{\partial x \partial y} = 12y$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (-12x)(12x - 36y^2) - (12y)^2 = -144x^2 + 432xy^2 - 144y^2$$

$D(0, 0) = 0$, so the test is inconclusive.

$D(1, 1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, 1) < 0$, so $f(x, y)$ has a relative maximum at $(1, 1)$.

$D(1, -1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, -1) < 0$, so $f(x, y)$ has a relative maximum at $(1, -1)$.

21. $f(x, y) = 2x^2 - x^4 - y^2$

$$\frac{\partial f}{\partial x} = 4x - 4x^3; \frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial^2 f}{\partial x^2} = 4 - 12x^2; \frac{\partial^2 f}{\partial y^2} = -2; \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (4 - 12x^2)(-2) - (0)^2 = -8 + 24x^2$$

$D(-1, 0) > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, 0) < 0$, so $f(x, y)$ has a relative maximum at $(-1, 0)$. $D(0, 0) < 0$, so $f(x, y)$ has neither a relative minimum nor a relative maximum at $(0, 0)$. $D(1, 0) > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, 0) < 0$, so $f(x, y)$ has a relative maximum at $(1, 0)$.

22. $f(x, y) = x^4 - 4xy + y^4$

$$\frac{\partial f}{\partial x} = 4x^3 - 4y; \frac{\partial f}{\partial y} = -4x + 4y^3$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2; \frac{\partial^2 f}{\partial y^2} = 12y^2; \frac{\partial^2 f}{\partial x \partial y} = -4$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 12x^2(12y^2) - (-4)^2 = 144x^2y^2 - 16$$

$D(0, 0) < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(0, 0)$.

$D(1, 1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, 1) > 0$, so $f(x, y)$ has a relative minimum at $(1, 1)$.

$D(-1, -1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, -1) > 0$, so $f(x, y)$ has a relative minimum at $(-1, -1)$.

23. $f(x, y) = ye^x - 3x - y + 5$

$$\frac{\partial f}{\partial x} = ye^x - 3; \frac{\partial f}{\partial y} = e^x - 1$$

$$\frac{\partial^2 f}{\partial x^2} = ye^x; \frac{\partial^2 f}{\partial y^2} = 0; \frac{\partial^2 f}{\partial x \partial y} = e^x$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = ye^x(0) - (e^x)^2 = -e^{2x}$$

$D(0, 3) < 0$, thus $f(x, y)$ has neither a maximum nor a minimum at $(0, 3)$.

24. $f(x, y) = \frac{1}{x} + \frac{1}{y} + xy$;

$\frac{\partial f}{\partial x} = \frac{-1}{x^2} + y; \frac{\partial f}{\partial y} = \frac{-1}{y^2} + x$

$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}; \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3}; \frac{\partial^2 f}{\partial x \partial y} = 1$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \frac{2}{x^3} \cdot \frac{2}{y^3} - 1^2 = \frac{4}{x^3 y^3} - 1$$

$D(1, 1) > 0$, $\frac{\partial^2 f}{\partial x^2}(1, 1) > 0$, so $f(x, y)$ has a relative minimum at (1, 1).

25. $f(x, y) = -5x^2 + 4xy - 17y^2 - 6x + 6y + 2$;

$\frac{\partial f}{\partial x} = -10x + 4y - 6;\ \frac{\partial f}{\partial y} = -34y + 4x + 6$

$\frac{\partial^2 f}{\partial x^2} = -10;\ \frac{\partial^2 f}{\partial y^2} = -34;\ \frac{\partial^2 f}{\partial x \partial y} = 4$

$$\left.\begin{array}{r} -10x + 4y - 6 = 0 \\ 4x - 34y + 6 = 0 \end{array}\right\} x = -\frac{5}{9},\ y = \frac{1}{9}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -10(-34) - 4^2 = 324$$

$D\left(-\frac{5}{9}, \frac{1}{9}\right) = -10(-34) - 4^2 = 324 > 0$ and $\frac{\partial^2 f}{\partial x^2}\left(-\frac{5}{9}, \frac{1}{9}\right) < 0$, so $f(x, y)$ has a relative maximum at $\left(-\frac{5}{9}, \frac{1}{9}\right)$.

26. $f(x, y) = -2x^2 + 6xy - 17y^2 - 4x + 6y$;

$\frac{\partial f}{\partial x} = -4x + 6y - 4;\ \frac{\partial f}{\partial y} = -34y + 6x + 6$

$\frac{\partial^2 f}{\partial x^2} = -4;\ \frac{\partial^2 f}{\partial y^2} = -34;\ \frac{\partial^2 f}{\partial x \partial y} = 6$

$$\left.\begin{array}{r} -4x + 6y - 4 = 0 \\ 6x - 34y + 6 = 0 \end{array}\right\} x = -1,\ y = 0$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -4(-34) - 6^2 = 100$$

$D(-1, 0) = -4(-34) - 6^2 = 100 > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, 0) < 0$, so $f(x, y)$ has a relative maximum at $(-1, 0)$.

27. $f(x, y) = 3x^2 + 8xy - 3y^2 + 2x + 6y$;

$\frac{\partial f}{\partial x} = 6x + 8y + 2;\ \frac{\partial f}{\partial y} = -6y + 8x + 6$

$\frac{\partial^2 f}{\partial x^2} = 6;\ \frac{\partial^2 f}{\partial y^2} = -6;\ \frac{\partial^2 f}{\partial x \partial y} = 8$

$$\left.\begin{array}{r} 6x + 8y + 2 = 0 \\ 8x - 6y + 6 = 0 \end{array}\right\} x = -\frac{3}{5},\ y = \frac{1}{5}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 6(-6) - 8^2 = -100$$

$D\left(-\frac{3}{5}, \frac{1}{5}\right) = 6(-6) - 8^2 = -100 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $\left(-\frac{3}{5}, \frac{1}{5}\right)$.

28. $f(x, y) = 8xy + 8y^2 - 2x + 2y - 1$;

$\frac{\partial f}{\partial x} = 8y - 2;\ \frac{\partial f}{\partial y} = 16y + 8x + 2$

$\frac{\partial^2 f}{\partial x^2} = 0;\ \frac{\partial^2 f}{\partial y^2} = 16;\ \frac{\partial^2 f}{\partial x \partial y} = 8$

$$\left.\begin{array}{r} 8y - 2 = 0 \\ 8x + 16y + 2 = 0 \end{array}\right\} x = -\frac{3}{4},\ y = \frac{1}{4}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0(16) - 8^2 = -64 < 0$$

The potential relative extreme point is $\left(-\frac{3}{4}, \frac{1}{4}\right)$. However, because $D(x, y) < 0$, it is neither a relative maximum nor a relative minimum.