Homework 27 Solutions Math 141

1. $f(x, y) = 5xy$

$\frac{\partial f}{\partial x} = 5(1)y = 5y;\ \ \frac{\partial f}{\partial y} = 5(1)x = 5x$

2. $f(x, y) = x^2 - y^2$

$\frac{\partial f}{\partial x} = 2x;\ \frac{\partial f}{\partial y} = -2y$

3. $f(x, y) = 2x^2 e^y$

$\frac{\partial f}{\partial x} = 2(2x)e^y = 4xe^y$

$\frac{\partial f}{\partial y} = 2x^2 e^y(1) = 2x^2 e^y$

4. $f(x, y) = xe^{xy}$

$\frac{\partial f}{\partial x} = xe^{xy}(y) + (1)e^{xy} = xe^{xy}y + e^{xy}$

$\frac{\partial f}{\partial y} = xe^{xy}(x) + (0)e^{xy} = x^2 e^{xy}$

5. $f(x, y) = \frac{x}{y} + \frac{y}{x} = xy^{-1} + x^{-1}y$

$\frac{\partial f}{\partial x} = \frac{1}{y} - \frac{y}{x^2};\ \ \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{1}{x}$

6. $f(x, y) = \frac{1}{x+y} = (x+y)^{-1}$

$\frac{\partial f}{\partial x} = -\frac{1}{(x+y)^2};\ \ \frac{\partial f}{\partial y} = -\frac{1}{(x+y)^2}$

7. $f(x, y) = (2x - y + 5)^2$

$\frac{\partial f}{\partial x} = 2(2x - y + 5)(2) = 4(2x - y + 5)$

$\frac{\partial f}{\partial y} = 2(2x - y + 5)(-1) = -2(2x - y + 5)$

8. $f(x, y) = \frac{e^x}{1+e^y}$

$\frac{\partial f}{\partial x} = \frac{e^x}{1+e^y};\ \frac{\partial f}{\partial y} = \frac{-e^x e^y}{(1+e^y)^2} = \frac{-e^{x+y}}{(1+e^y)^2}$

9. $f(x, y) = xe^{x^2y^2}$

$$\begin{aligned}\frac{\partial f}{\partial x} &= x\left(\frac{\partial}{\partial x}e^{x^2y^2}\right) + e^{x^2y^2}\left(\frac{\partial}{\partial x}x\right)\\ &= x\left(e^{x^2y^2}\frac{\partial}{\partial x}x^2y^2\right) + e^{x^2y^2}\\ &= 2x^2y^2e^{x^2y^2} + e^{x^2y^2}\\ &= e^{x^2y^2}\left(2x^2y^2 + 1\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= x\left(\frac{\partial}{\partial y}e^{x^2y^2}\right) = xe^{x^2y^2}\left(\frac{\partial}{\partial y}x^2y^2\right)\\ &= xe^{x^2y^2}\left(2x^2y\right) = 2x^3ye^{x^2y^2}\end{aligned}$$

10. $f(x, y) = \ln(xy)$

$\frac{\partial f}{\partial x} = \frac{1}{xy}\cdot y = \frac{1}{x}$

$\frac{\partial f}{\partial y} = \frac{1}{xy}\cdot x = \frac{1}{y}$

11. $f(x, y) = \frac{x-y}{x+y}$

$\frac{\partial f}{\partial x} = \frac{1(x+y) - (x-y)(1)}{(x+y)^2} = \frac{2y}{(x+y)^2}$

$\frac{\partial f}{\partial y} = \frac{(-1)(x+y) - (x-y)(1)}{(x+y)^2} = -\frac{2x}{(x+y)^2}$

12. $f(x, y) = \sqrt{x^2 + y^2} = \left(x^2 + y^2\right)^{1/2}$

$\frac{\partial f}{\partial x} = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$

$\frac{\partial f}{\partial y} = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$

13. $f(L, K) = 3\sqrt{LK}$

$\frac{\partial f}{\partial L} = 3\left(\frac{1}{2}\right)(KL)^{-1/2}(K) = \frac{3}{2}\sqrt{\frac{K}{L}}$

14. $f(p, q) = 1 - p(1+q) = 1 - p - pq$

$\frac{\partial f}{\partial q} = -p;\ \ \frac{\partial f}{\partial p} = -1 - q$

15. $f(x, y, z) = \frac{(1 + x^2y)}{z} = z^{-1} + x^2yz^{-1}$

$\frac{\partial f}{\partial x} = 0 + 2xyz^{-1} = \frac{2xy}{z}$

$\frac{\partial f}{\partial y} = 0 + x^2z^{-1} = \frac{x^2}{z}$

$$\begin{aligned}\frac{\partial f}{\partial z} &= -z^{-2} + (-z^{-2}x^2y) = -\frac{1}{z^2} - \frac{x^2y}{z^2}\\ &= -\frac{1 + x^2y}{z^2}\end{aligned}$$

16. $f(x, y, z) = ze^{x/y} = ze^{xy^{-1}}$

$\frac{\partial f}{\partial x} = ze^{xy^{-1}}y^{-1} = \frac{ze^{x/y}}{y}$

$\frac{\partial f}{\partial y} = ze^{xy^{-1}}\left(-xy^{-2}\right) = \frac{-xze^{x/y}}{y^2}$

$\frac{\partial f}{\partial z} = e^{x/y}$

17. $f(x, y, z) = xze^{yz}$

$$\frac{\partial f}{\partial x} = (1)ze^{yz} = ze^{yz}$$

$$\frac{\partial f}{\partial y} = e^{yz}(z)xz = xz^2e^{yz}$$

$$\frac{\partial f}{\partial z} = (1)xe^{yz} + e^{yz}(y)xz = xe^{yz}(1 + yz)$$

18. $f(x, y, z) = \frac{xy}{z} = xyz^{-1}$

$$\frac{\partial f}{\partial x} = \frac{y}{z}; \quad \frac{\partial f}{\partial y} = \frac{x}{z}$$

$$\frac{\partial f}{\partial z} = -xyz^{-2} = -\frac{xy}{z^2}$$

19. $f(x, y) = x^2 + 2xy + y^2 + 3x + 5y$

$$\frac{\partial f}{\partial x} = 2x + 2(y) + 0 + 3 + 0 = 2x + 2y + 3$$

$$\frac{\partial f}{\partial x}(2, -3) = 2(2) + 2(-3) + 3 = 1$$

$$\frac{\partial f}{\partial y} = 0 + 2x(1) + 2y + 0 + 5 = 2x + 2y + 5$$

$$\frac{\partial f}{\partial y}(2, -3) = 2(2) + 2(-3) + 5 = 3$$

20. $f(x, y) = \left(x + y^2\right)^3$

$$\frac{\partial f}{\partial x} = 3\left(x + y^2\right)^2(1) = 3\left(x + y^2\right)^2$$

$$\frac{\partial f}{\partial x}(1, 2) = 3(1 + 2^2)^2 = 75$$

$$\frac{\partial f}{dy} = 3\left(1 + y^2\right)^2(2y) = 6y\left(1 + y^2\right)^2$$

$$\frac{\partial f}{\partial y}(1, 2) = 6(2)\left(1 + 2^2\right)^2 = 300$$

21. $f(x, y) = xy^2 + 5$

$$\frac{\partial f}{\partial y} = 2xy$$

$$\frac{\partial f}{\partial y}(2, -1) = 2(2)(-1) = -4$$

This means that if x is kept constant at 2 and y is allowed to vary near −1, then $f(x, y)$ changes at a rate of −4 times the change in y.

22. $f(x, y) = \frac{x}{y - 6}$

$$\frac{\partial f}{\partial y} = -\frac{x}{(y - 6)^2}$$

$$\frac{\partial f}{\partial y}(2, 1) = -\frac{2}{(1 - 6)^2} = -\frac{2}{25}$$

This means that if x is kept constant at 2 and y is allowed to vary near 1, then $f(x, y)$ changes at a rate of $-\frac{2}{25}$ times the change in y.

23. $f(x, y) = x^3y + 2xy^2$

$$\frac{\partial f}{\partial x} = 3x^2y + 2y^2 \Rightarrow \frac{\partial^2 f}{\partial x^2} = 6xy$$

$$\frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 4y$$

$$\frac{\partial f}{\partial y} = x^3 + 4xy \Rightarrow \frac{\partial^2 f}{\partial y^2} = 4x$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 4y$$

24. $f(x, y) = xe^y + x^4y + y^3$

$$\frac{\partial f}{\partial x} = e^y + 4x^3y + 0 = e^y + 4x^3y$$

$$\frac{\partial^2 f}{\partial x^2} = 0 + 12x^2y = 12x^2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = e^y(1) + 4x^3 = e^y + 4x^3$$

$$\frac{\partial f}{\partial y} = xe^y(1) + x^4 + 3y^2 = xe^y + x^4 + 3y^2$$

$$\frac{\partial^2 f}{\partial y^2} = xe^y(1) + 0 + 6y = xe^y + 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^y + 4x^3 + 0 = e^y + 4x^3$$

25. $f(x, y) = 200\sqrt{6x^2 + y^2}$

a. $\frac{\partial f}{\partial x}$ is the marginal productivity of labor.

$$\frac{\partial f}{\partial x} = 200\left(\frac{1}{2}\right)(6x^2 + y^2)^{-1/2}(12x)$$

$$= 1200x(6x^2 + y^2)^{-1/2} = \frac{1200x}{\sqrt{6x^2 + y^2}}$$

(*continued*)

When $x = 10$ and $y = 5$, the marginal productivity of labor is

$$\frac{\partial f}{\partial x}(10,\ 5) = \frac{1200(10)}{\sqrt{6(10)^2 + 5^2}} = 480.$$

$\frac{\partial f}{\partial y}$ is the marginal productivity of capital.

$$\frac{\partial f}{\partial y} = 200\left(\frac{1}{2}\right)(6x^2 + y^2)^{-1/2}(2y)$$

$$= 200y(6x^2 + y^2)^{-1/2} = \frac{200y}{\sqrt{6x^2 + y^2}}$$

When $x = 10$ and $y = 5$, the marginal productivity of capital is

$$\frac{\partial f}{\partial y}(10,\ 5) = \frac{200(5)}{\sqrt{6(10)^2 + 5^2}} = 40.$$

b. $f(10 + h, 5) - f(10, 5) \approx \frac{\partial f}{\partial x}(10,\ 5) \cdot h$
$= 480h$

c. Using part b, if $h = -.5$ then $f(9.5, 5) \approx 480(-.5) = -240$. So, if capital is fixed at 5 units and labor decreased by .5 unit from 10 to 9.5 units, the number of goods produced will decrease by approximately 240 units.

26. $f(x, y) = 300x^{2/3}y^{1/3}$ is the productivity of a country, where x and y are the amounts of labor and capital.

a. $\frac{\partial f}{\partial x}$ is the marginal productivity of labor.

$$\frac{\partial f}{\partial x} = 300\left(\frac{2}{3}\right)x^{-1/3}y^{1/3} = \frac{200\sqrt[3]{y}}{\sqrt[3]{x}}$$

When $x = 125$ and $y = 64$, the marginal productivity of labor is

$$\frac{\partial f}{\partial x}(125,\ 64) = \frac{200\sqrt[3]{64}}{\sqrt[3]{125}} = 160.$$

$\frac{\partial f}{\partial y}$ is the marginal productivity of capital.

$$\frac{\partial f}{\partial y} = 300x^{2/3}\left(\frac{1}{3}\right)y^{-2/3} = \frac{100\sqrt[3]{x^2}}{\sqrt[3]{y^2}}$$

When $x = 125$ and $y = 64$, the marginal productivity of capital is

$$\frac{\partial f}{\partial y}(125,\ 64) = \frac{100\sqrt[3]{125^2}}{\sqrt[3]{64^2}} = 156.25.$$

b. $f(125, 66) = f(125, 64 + 66)$

$$\approx \frac{\partial f}{\partial y}(125,\ 64) \cdot 2$$

$$= 156.25 \cdot 2 = 312.5$$

So if labor is fixed at 125 units and capital is increased from 64 to 66 units, then productivity increases by 312.5 units.

c. $f(124, 64) = f(125 - 1, 64)$

$$\approx \frac{\partial f}{\partial x}(125,\ 64)(-1)$$

$$= 160(-1) = -160$$

Thus, if capital is fixed at 64 units and labor is decreased by one unit, then productivity will decrease by 160 units.

27. As the price of a bus ride increases, fewer people will ride the bus if the train fare remains constant. An increase in train ticket prices, coupled with constant bus fare, should cause more people to ride the bus.

28. $g(p_1, p_2)$ is the number of people who will take the train when p_1 is the price of the bus ride and p_2 is the price of the train ride.

$\frac{\partial g}{\partial p_1}$ is positive (an increase in bus fare would mean more people would take the train).

$\frac{\partial g}{\partial p_2}$ is negative (an increase in the train fare would mean fewer people taking the train).

29. If the average price of MP3 players remains constant and the average price of audio files increases, people will purchase fewer MP3 players. An increase in the average price of the MP3 players, coupled with constant audio file prices, should cause a decline in the number of audio files purchased.

30. When the gasoline price is constant, an increase in the price of the car will decrease the demand for the car. If the price of the car is constant and the price of the gasoline increases, the demand for the car will decrease.

31. $V = .08\left(\frac{T}{P}\right) \Rightarrow \frac{\partial V}{\partial P} = \frac{-.08T}{P^2}$

When $P = 20$, $T = 300$,

$\frac{\partial V}{\partial P} = \frac{-.08(300)}{400} = -.06$.

At this level, increasing the pressure by one unit will decrease the volume by approximately .06 unit.

$\frac{\partial V}{\partial T} = \frac{.08}{P}$

When $P = 20$, $T = 300$, $\frac{\partial V}{\partial T} = \frac{.08}{20} = .004$.

At this level, increasing the temperature by one unit will increase the volume by approximately .004 unit.

32. Assuming $m, p, r, s > 0$, all first partial derivatives are positive except

$\frac{\partial f}{\partial p} \approx -.769 m^{1.136} r^{0.914} s^{0.816} p^{-1.727} < 0.$

Thus increases in aggregate income, retail prices of the other goods or the strength of the beer (holding the other quantities constant) should cause an increase in the amount of beer consumed; while an increase in the price of beer itself should cause the amount consumed to decrease.

33. Assuming $m, p, r > 0$, $\frac{\partial f}{\partial m} > 0, \frac{\partial f}{\partial r} > 0$ and

$\frac{\partial f}{\partial p} \approx -1.187 m^{.595} r^{.922} p^{-1.543} < 0$.

Thus increases in aggregate income or retail prices of other goods (holding the other quantities constant) should cause an increase in the amount of food consumed; while an increase in the price of the food itself should cause the amount consumed to decrease.

34. $f(x, y) = 60x^{3/4}y^{1/4} \Rightarrow \frac{\partial f}{\partial x} = 45y^{1/4}x^{-1/4}$

$\frac{\partial f}{\partial y} = 15x^{3/4}y^{-3/4}$

$$\begin{aligned} f(a, b) &= 60a^{3/4}b^{1/4} \\ &= a\left[45b^{1/4}a^{-1/4}\right] + b\left[15a^{3/4}b^{-3/4}\right] \\ &= a\left[\frac{\partial f}{\partial x}(a, b)\right] + b\left[\frac{\partial f}{\partial y}(a, b)\right] \end{aligned}$$

35. $f(x, y) = 60x^{3/4}y^{1/4}$

$\frac{\partial f}{\partial x} = 45y^{1/4}x^{-1/4} \Rightarrow \frac{\partial^2 f}{\partial x^2} = -\frac{45}{4}y^{1/4}x^{-5/4} < 0$

for all $x, y > 0$. The fact that $\frac{\partial^2 f}{\partial x^2} < 0$ confirms the *law of diminishing returns*, which says that as additional units of a given productive input are added (holding other factors constant) production increases at a decreasing rate. In other words, the marginal productivity of labor is decreasing.

36. $f(x, y) = 60x^{3/4}y^{1/4}$

$\frac{\partial f}{\partial y} = 15x^{3/4}y^{-3/4} \Rightarrow$ for all $x, y > 0$.

$\frac{\partial^2 f}{\partial y^2} = -\frac{45}{4}x^{3/4}y^{-7/4} < 0$

The fact that $\frac{\partial^2 f}{\partial y^2} < 0$ confirms the law of diminishing returns, which says that as additional units of a given productive input are added (holding other factors constant) production increases at a decreasing rate.

37. $f(x, y) = 3x^2 + 2xy + 5y$

$$\begin{aligned} &f(1+h, 4) - f(1, 4) \\ &= \left[3(1+h)^2 + 2(1+h)(4) + 5(4)\right] \\ &\quad - \left[3(1)^2 + 2(1)(4) + 5(4)\right] \\ &= 3h^2 + 14h \end{aligned}$$

38. $A = (.007)W^{0.425}H^{0.725}$

$\frac{\partial A}{\partial W} = (.002975)W^{-0.575}H^{0.725}$

When $W = 54$, $H = 165$,

$\frac{\partial A}{\partial W} = .002975(54)^{-0.575}(165)^{0.725} \approx .01216$

If a person weighing 54 kg who is 165 cm tall increases his weight by 1 kg, the surface area of his body will increase by about $.012 \text{ cm}^2$.

$\frac{\partial A}{\partial H} = (.005075)W^{0.425}H^{-0.275}$

When $W = 54$, $H = 165$, $\frac{\partial A}{\partial H} \approx .0067904$. If a person as above increases his height by 1 cm, his body surface will increase by approximately $.0068 \text{ m}^2$.