Section 5.3 Binomial Trials
Math 141

Main ideas

Binomial trial: an experiment with two outcomes: "success" or "failure".

In n trials, the probability of:
k successes, each with probability p
$n-k$ failures, each with probability $1-p$
is $\boldsymbol{C(n,k)} \cdot \boldsymbol{p^k(1-p)^{n-k}} = \boldsymbol{\binom{n}{k} p^k(1-p)^{n-k}} = \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$

Number of ways to have exactly k successes	Probability of a specific outcome that has k successes and $n-k$ failures

Problems

1. 70% free throw shooter. Shoot 2 shots.
   Assume the two shots are independent.
   Pr(make 1, make 1) = $(.7)(.7) = .49$
   Pr(miss 1, miss 1) = $(.3)(.3) = .09$
   Pr(make 1, miss 1) = $(.7)(.3) = .21$
   Pr(miss 1, make 1) = $(.3)(.7) = .21$

   Add up to 1.

2. 70% free thrown shooter. Shoot 10 shots.
   Pr(make first 6, miss next 4) = $(.7)^6(.3)^4$
   Pr(miss first 4, make next 6) = $(.3)^4(.7)^6$
   Pr(make 1, miss 1, make 3, miss 2, make 2, miss 1) = $(.7)(.3)(.7)^3(.3)^2(.7)^2(.3)$
   Pr(make 6 of the 10, miss the other 4, <u>in a specific order</u>) = $(.7)^6(.3)^4$
   There are $C(10,6) = 210$ ways to make 6 and miss 4 shots if shooting 10 total.
   Each has probability of $(.7)^6(.3)^4 \approx .000953$ of occurring.

   So the probability of making 6 of 10 shots (in any order) is

   $.000953 + .000953 + \ldots.. + .000953$ 210 times.

   Or we could simply find $210 \cdot .000953 \approx .2001$.

   That is, C(10 , 6) $\cdot$ ($.7$)^6($.3$)^4 $\approx .2001$.

3. 70% free throw shooter. Shoot 10 shots. X is number of shots made.

k	Pr(X = k)	
0	$C(10,0)(.7)^0(.3)^{10} = .00006$	$\leftarrow (.3)^{10}$
1	$C(10,1)(.7)^1(.3)^9 = .0001$	
2	$C(10,2)(.7)^2(.3)^8 = .0014$	
3	$C(10,3)(.7)^3(.3)^7 = .0090$	
4	$C(10,4)(.7)^4(.3)^6 = .0368$	
5	$C(10,5)(.7)^5(.3)^5 = .1029$	
6	$C(10,6)(.7)^6(.3)^4 = .2001$	
7	$C(10,7)(.7)^7(.3)^3 = .2668$	
8	$C(10,8)(.7)^8(.3)^2 = .2335$	
9	$C(10,9)(.7)^9(.3)^1 = .1211$	
10	$C(10,10)(.7)^{10}(.3)^0 = .0282$	$\leftarrow (.7)^{10}$

4. Generalizations in free throw shooting:

Shots taken	**Prob. of making one shot**	**Shots to make**	**Probability of making k shots**
10	.70	6	$C(10,6)(.7)^6(.3)^4$
10	.70	**k**	$C(10,k)(.7)^k(.3)^{10-k}$
n	.70	k	$C(n,k)(.7)^k(.3)^{n-k}$
n	**p**	k	$C(n,k)\,p^k(1-p)^{n-k}$

5. 70% free throw shooter. Shoot 5 shots. Which is higher of:

Pr(make 3 of 5 shots) = $C(5,3)\cdot(.7)^3(.3)^2 = .3087$

Pr(make 4 of 5 shots) = $C(5,4)\cdot(.7)^4(.3)^1 = .3602$

6. What if what we call "success" someone else calls "failure," and conversely.

If "success" is missing a shot, then: Pr(miss 4 of 10) = $C(10,4)(.3)^4(.7)^6$

Compare to result found earlier: Pr(make 6 of 10) = $C(10,6)(.7)^6(.3)^4$

7. Slightly paradoxical observation. The more shots you take:
   - The *less* likely it is you will make *exactly* the percentage that is your FT percentage.
   - The *more* likely it is you will make a percentage that is *in the range* of your FT percentage.

   Probability of making exactly 70% of your shots; let n be the number of shots taken.

n	70% of n	Pr(make exactly 70% of the shots)
10	7	$C(10,7)(.7)^{7}(.3)^{3} = .2668$
20	14	$C(20,14)(.7)^{14}(.3)^{6} = .1916$
30	21	$C(30,21)(.7)^{21}(.3)^{9} = .1573$
40	28	$C(40,28)(.7)^{28}(.3)^{12} = .1366$
50	35	$C(50,35)(.7)^{35}(.3)^{15} = .1223$
$\downarrow$ ∞		$\downarrow$ 0

   Probability of making between 60 and 80% of your shots:

   If shooting 10 shots, make 6 or 7 or 8 of the 10:
   $C(10, 6)\,(.70)^6(.30)^4 + C(10, 7)\,(.70)^7(.30)^3 + C(10, 8)\,(.70)^8(.30)^2 = .7004$

   If shooting 20 shots, make either 12 or 13 … or 16 of the 20:
   $C(20, 12)\,(.70)^{12}(.30)^8 + C(20, 13)\,(.70)^{13}(.30)^7 + \ldots + C(20, 16)\,(.70)^{16}(.30)^4 = .7796$

   If shooting 30 shots, make either 18 or 19 or … or 24 of the 30:
   $C(30, 18)\,(.70)^{18}(.30)^{12} + C(30, 19)\,(.70)^{19}(.30)^{11} + \ldots + C(30, 24)\,(.70)^{24}(.30)^6 = .8389$

   If shooting 40 shots, make either 24 or 25 or … or 32 of the 40:
   $C(40, 24)\,(.70)^{24}(.30)^{16} + C(40, 25)\,(.70)^{25}(.30)^{15} + \ldots + C(40, 32)\,(.70)^{32}(.30)^8 = .8814$

   If shooting 50 shots, make either 30 or 31 or … or 40 of the 50:
   $C(50, 30)\,(.70)^{30}(.30)^{20} + C(50, 31)\,(.70)^{31}(.30)^{19} + \ldots + C(50, 40)\,(.70)^{40}(.30)^{10} = .9120$

8. Suppose that 40% of all cars are white. Of the next 10 cars you see (of random colors), find the probabilities that:

   3 are white: $C(10,3)\cdot(.4)^3(.6)^7 = .2150$

   4 are white: $C(10,4)\cdot(.4)^4(.6)^6 = .2508$

   5 are white: $C(10,5)\cdot(.4)^5(.6)^5 = .2007$

9. 12 jurors. Each will vote guilty or not guilty. In order to convict, they need 10 or more to vote guilty. Based on the evidence presented, it is estimated that the probability of an individual juror voting guilty is .80. Let X be the number who vote guilty.

   $\Pr(X \geq 10) = \Pr(X = 10) + \Pr(X = 11) + \Pr(X = 12)$
   $= C(12,10)(.8)^{10}(.2)^2 + C(12,11)(.8)^{11}(.2)^1$
   $+ C(12,12)(.8)^{12}(.2)^0$
   $= .2835 + .2062 + .0687$
   $= .5584$

10. Roll a single die 12 times. What is the probability of getting a 3:

0 times: $C(12,0)\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{12} = .1122$

1 time: $C(12,1)\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{11} = .2692$

2 times: $C(12,2)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{10} = .2961$

3 times: $C(12,3)\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{9} = .1974$

12 times: $C(12,12)\left(\frac{1}{6}\right)^{12}\left(\frac{5}{6}\right)^{0} = .000000000459$

Compare to: $\dfrac{C(12,3)\cdot 1^3 \cdot 5^9}{6^{12}}$

11. The probability that a 21 year old woman will live to be 100 years old (or more) is 2.17%. What is the probability that of 46 (which ≈ 1/.0217) 21 year old women:

None of the women will live to be 100?

$C(46,0)\cdot(.0217)^0(.9783)^{46} = .3645$

At least one will live to be 100?

$Pr(\geq 1 \quad 100) = 1 - Pr(0 \quad 100)$

$= 1 - .3645 = .6355$

Exactly one will live to be 100 (and the other 45 will not)?

$C(46,1)\cdot(.0217)^1(.9783)^{45} = .3719$

At least two will live to be 100?

$Pr(\geq 2 \quad 100) = 1 - Pr(< 2 \quad 100)$

$= 1 - [.3645 + .3719] = .2636$

Exactly two will live to be 100 (and the other 44 will not)?

$C(46,2)\cdot(.0217)^2(.9783)^{44} = .1856$

12. Suppose that in a family both the father and mother have one dominant **A** gene and one recessive **a** gene. What is the probability that at least one of their three kids will have the double recessive gene pair **aa**?

	A	a
A	AA	Aa
a	aA	aa

Child's genes	Prob.
AA	1/4
Aa	2/4
aa	1/4

Let X = # kids with aa gene pair

$$Pr(X \geq 1) = Pr(X=1) + Pr(X=2) + Pr(X=3)$$

$$= C(3,1)\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^2 + C(3,2)\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^1 + C(3,3)\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^0$$

$$= \frac{27}{64} + \frac{9}{64} + \frac{1}{64} = \frac{37}{64}$$

OR:

$$Pr(X \geq 1) = 1 - Pr(X=0)$$

$$= 1 - C(3,0)\left(\tfrac{1}{4}\right)^0\left(\tfrac{3}{4}\right)^3$$

$$= 1 - \left(\tfrac{3}{4}\right)^3 = 1 - \frac{27}{64} = \frac{37}{64}$$

13. A family chosen at random has <u>four</u> children. Assume having boys and girls is equally likely. What is the probability that there are two boys and two girls?

$$C(4,2)\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^2 = \frac{C(4,2)}{2^4} = \frac{6}{16}$$

What is the probability that there are three of one gender and one of the other?

3B/1G: $C(4,3)\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^1 = \frac{C(4,3)}{2^4} = \frac{4}{16}$

1B/3G: Also $\frac{4}{16}$

$\left.\begin{matrix}\frac{4}{16} \\ \frac{4}{16}\end{matrix}\right\} \frac{8}{16}$

What is the probability that all four kids are of the same gender?

4B/0G: $C(4,4)\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^0 = \frac{1}{16}$

0B/4G: Also $\frac{1}{16}$

$\left.\begin{matrix}\frac{1}{16} \\ \frac{1}{16}\end{matrix}\right\} \frac{2}{16}$

Notice:

$$\frac{6}{16} + \frac{8}{16} + \frac{2}{16} = \frac{16}{16}.$$