Section 4.4 Conditional Probabilities and Independence

Math 141

Main ideas

Two types of probabilities given events A and B and sample space S:

Probability of A:

$$\Pr(A) = \frac{n(A)}{n(S)}$$

Probability of A, given that B is true (or has occurred):

$$\mathbf{Pr}(\boldsymbol{A}|\boldsymbol{B}) = \frac{\mathbf{n}(\boldsymbol{A}\cap\boldsymbol{B})}{\mathbf{n}\,(\boldsymbol{B})}$$

$$\mathbf{Pr}(\boldsymbol{A}|\boldsymbol{B}) = \frac{\mathbf{Pr}\,(\boldsymbol{A}\cap\boldsymbol{B})}{\mathbf{Pr}\,(\boldsymbol{B})} \quad \text{thus} \quad \mathbf{Pr}(\mathbf{A}\cap\mathbf{B}) = \mathbf{Pr}(\boldsymbol{B})\,\mathbf{Pr}\,(\boldsymbol{A}|\boldsymbol{B})$$

$$\mathbf{Pr}(\boldsymbol{B}|\boldsymbol{A}) = \frac{\mathbf{Pr}\,(\boldsymbol{A}\cap\boldsymbol{B})}{\mathbf{Pr}\,(\boldsymbol{A})} \quad \text{thus} \quad \mathbf{Pr}\,(\mathbf{A}\cap\mathbf{B}) = \mathbf{Pr}(\boldsymbol{A})\,\mathbf{Pr}\,(\boldsymbol{B}|\boldsymbol{A})$$

If A is independent of B (and similarly if B is independent of A):

$$\Pr(A|B) = \Pr(A)$$

$$\Pr(A\cap B) = \Pr(B)\cdot\Pr(A|B) = \Pr(B)\cdot\Pr(A) = \Pr(A)\cdot\Pr(B)$$

Observation:

Suppose that A is independent of B. Then Pr(A ∩ B) = Pr(A)·Pr(B|A) = Pr(A)·Pr(B). Then Pr(B)·Pr(A|B) = Pr(A ∩ B) = Pr(A)·Pr(B), which means that Pr(A|B) = Pr(A), i.e. B is independent of A. In other words, A and B are independent of each other.

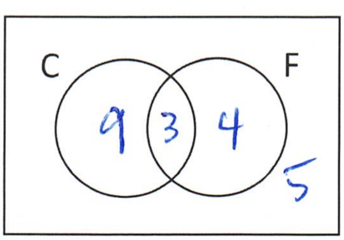
Observations

→ $Pr(C|F) \neq Pr(C)$ → $Pr(C|F) \neq Pr(F|C)$

→ $Pr(F|C) \neq Pr(F)$ → $Pr(C|F) + Pr(C'|F) = 1$

Problems

1. Students in our class: from **C**alifornia (or not) and **F**emale (or not):



	C	C'
F	3	4
F'	9	5

Pr(C) = 12/21 Pr(C') = 9/21

Pr(F) = 7/21 Pr(F') = 14/21

Pr(C|F) = 3/7 Pr(C'|F) = 4/7

Pr(C|F') = 9/14 Pr(C'|F')= 5/14

Pr(F|C) = 3/12 Pr(F'|C) = 9/12

Pr(F|C')= 4/9 Pr(F'|C') = 5/9

2. Roll 2 dice. We're interested in the sum.
   E is event that sum is odd.
   F is event that sum is ≤ 7.

Sum of 2 dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$\Pr(E) =$ 13/36

$\Pr(F) =$ 21/36

$\Pr(E \cap F) =$ 12/36

$\Pr(E|F) =$ 12/21

$\Pr(F|E) =$ 12/18

3. Suppose there is a bin with 4 green and 6 blue balls. Choose two balls.
   Consider the events: G1 = "first ball is green" B1 = "first ball card is blue"
   G2 = "second ball is green" B2 = "second ball is blue"

First find:

Pr(both balls are green) = $C(4,2) \cdot C(6,0) / C(10,2) = 12/90$

Pr(both balls are blue) = $C(4,0) \cdot C(6,2) / C(10,2) = 30/90$

Pr(one ball is green, one ball is blue) = $C(4,1) \cdot C(6,1) / C(10,2) = 48/90$

Next find:

Pr(G1) = 4/10

Pr(B1) = 6/10

Pr(G2 | G1) = 3/9

Pr(B2 | G1) = 6/9

Pr(G2 | B1) = 4/9

Pr(B2 | B1) = 5/9

Pr(G1 ∩ G2) = $\Pr(G1) \cdot \Pr(G2 | G1) = \frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90}$

Pr(B1 ∩ B2) = $\Pr(B1) \cdot \Pr(B2 | B1) = \frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90}$

Pr(G1 ∩ B2) = $\Pr(G1) \cdot \Pr(B2 | G1) = \frac{4}{10} \cdot \frac{6}{9} = \frac{24}{90}$

Pr(B1 ∩ G2) = $\Pr(B1) \cdot \Pr(G2 | B1) = \frac{6}{10} \cdot \frac{4}{9} = \frac{24}{90}$

$\frac{48}{90}$

4. Consider the information in the table at right.

		Political party			
		D	R	I	Total
Gender	M	400	700	300	1400
	F	600	300	200	1100
	Total	1000	1000	500	2500

Pr(R) = $\frac{1000}{2500} = .40$

Pr(F) = $\frac{1100}{2500} = .44$

Pr(R|F) = $\frac{300}{1100} \approx .27$ (which $\neq .40$)

Pr(F|R) = $\frac{300}{1000} = .30$ (which $\neq .44$)

Pr(R $\cap$ F) = $\frac{300}{2500} = .12$ (which $\neq (.40)(.44)$)

Are "being Republican" and "being female" independent? No.

What exactly does this mean?

Political preference and gender are related

5. Suppose that Pr(E) = 0.3, Pr(F) = 0.4, Pr(E $\cap$ F) = 0.2. Are E and F independent? No

Pr(E|F) = $\frac{Pr(E \cap F)}{Pr(F)} = \frac{.2}{.4} = .5$ $(\neq Pr(E))$

Pr(F|E) = $\frac{Pr(F \cap E)}{Pr(E)} = \frac{.2}{.3} \approx .67$ $(\neq Pr(F))$

Also note that Pr(E $\cap$ F) $\neq$ Pr(E)·Pr(F).

6. Suppose that Pr(E) = 0.5, Pr(F) = 0.4, Pr(E $\cap$ F) = 0.2. Are E and F are independent? Yes.

Pr(E|F) = $\frac{Pr(E \cap F)}{Pr(F)} = \frac{.2}{.4} = .5$ $(= Pr(E))$

Pr(F|E) = $\frac{Pr(F \cap E)}{Pr(E)} = \frac{.2}{.5} = .4$ $(= Pr(F))$

Also note that Pr(E $\cap$ F) = Pr(E)·Pr(F).

7. A hospital uses two tests to classify blood. Every blood sample is subjected to both tests.
   - C1 The first test correctly identifies blood types with probability 0.70.
   - C2 The second test correctly identifies blood type with probability 0.80.
   - The probability that at least one of the tests correctly identifies the blood type is 0.90.

   Find the probability that both tests correctly identify the blood type.

$$Pr(C1 \text{ or } C2) = Pr(C1) + Pr(C2) - Pr(C1 \text{ and } C2)$$

$$.90 \quad .70 \quad .80$$

$$.70 + .80 - .90 = .60$$

   Determine the probability that the first test is correct given that the second test is correct.

$$Pr(C1 | C2) = \frac{Pr(C1 \text{ and } C2)}{Pr(C2)} = \frac{.60}{.80} = .75 \; (> Pr(C1))$$

   Determine the probability that the second test is correct given that the first test is correct.

$$Pr(C2 | C1) = \frac{Pr(C2 \text{ and } C1)}{Pr(C1)} = \frac{.60}{.70} \approx .86 \; (> Pr(C2))$$

   Are the events "Test 1 correctly identifies the blood type" and "Test 2 correctly identifies the blood type" independent?

   No. $Pr(C1|C2) \neq Pr(C1)$
   $Pr(C2|C1) \neq Pr(C2)$

   One being true makes the other more likely.

8. Suppose we know what percentage of spouses within married couples smoke:

   Pr(Woman Smokes) = .20

   Pr(Man Smokes) = .30

   Pr(Both Smokes) = .10

   Is one spouse smoking independent of other smoking, or is there dependency? Yes.

   Pr(WS|MS) = $\frac{Pr(WS \text{ and } MS)}{Pr(MS)} = \frac{.10}{.30} \approx .33 > Pr(WS)$

   Compare this to Pr(WS) = .20. So the man smoking makes it more likely that the woman also smokes. The same is true in reverse:

   Pr(MS|WS) = $\frac{.10}{.20} = .5$ which > Pr(MS) by the same proportion.

M = Man lives   W = Woman lives

9. Consider an older married couple. Based on their demographics (age, smoke or not, drink or not, etc.), the probability that the husband will be alive ten years from now is 0.90 and the probability that the wife will be alive ten years from now is 0.95. *Assuming that the outcomes (of living and dying) for each are independent*, what is the probability that at least one of the two will be alive in ten years?

$$\left.\begin{aligned} Pr(M \text{ and } W) &= (.90)(.95) = .855 \\ Pr(M \text{ and } W') &= (.90)(.05) = .045 \end{aligned}\right\} .90$$

$$Pr(M' \text{ and } W) = (.10)(.95) = .095$$

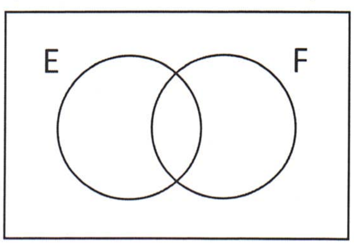
$$Pr(M' \text{ and } W') = (.10)(.05) = .005$$

.95

Sum of the four is 1.

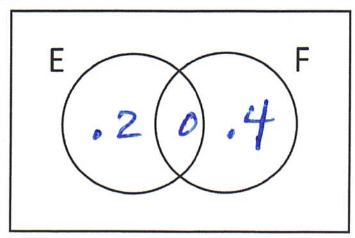
10. Independence vs. mutually exclusivity:

Independent and mutually exclusive.



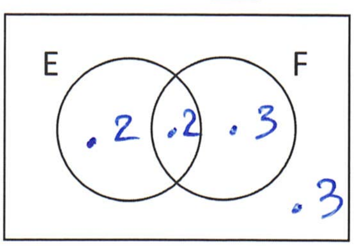
Not possible

<u>Not</u> independent and mutually exclusive.

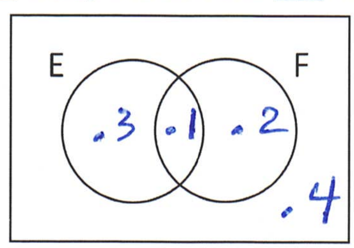


$$Pr(E|F) = \frac{0}{.4} = 0 \neq Pr(E)$$

Independent and <u>not</u> mutually exclusive.



<u>Not</u> independent and <u>not</u> mutually exclusive.



$$Pr(E|F) = \frac{.2}{.5} = .4 = Pr(E)$$

$$Pr(F|E) = \frac{.2}{.4} = .5 = Pr(F)$$

$$Pr(E \cap F) = Pr(E) \cdot Pr(F)$$

$$.2 \qquad .4 \qquad .5$$

$$Pr(E|F) = \frac{.1}{.3} \approx .33 \neq Pr(E)$$

$$Pr(F|E) = \frac{.1}{.4} = .25 \neq Pr(F)$$

$$Pr(E \cap F) \neq Pr(E) \cdot Pr(F)$$

$$.1 \qquad .4 \qquad .3$$