Section 3.6/3.8 Further Counting Problems/Multinomial Partitions

Math 141

Main ideas

Multiplication principle: if there are t tasks with $m_1,\ m_2, \dots, m_t$ choices, then there are $m_1 \cdot m_2 \cdot \cdots \cdot m_t$ ways to accomplish the t tasks.

Permutation: $P(n,r) = \frac{n!}{(n-r)!} = n \cdot (n-1) \cdot \cdots \cdot (n-r+1)$.

Combination—choose r items from n: $C(n,r) = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$.

Alternative view of combinations: divide n items into two groups (for example, those we keep, those we don't) of sizes r and $n-r$, that is, $\binom{n}{r,\ n-r} = \frac{n!}{r! \cdot (n-r)!}$.

Multinomial partition: the number of ways to divide n items into m groups of sizes $n_1, n_2, \dots, n_m$ (where $n_1 + n_2 + \cdots + n_m = n$) is $\binom{n}{n_1, n_2, \dots, n_m} = \frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_m!}$.

There are $r! = r \cdot (r-1) \cdot (r-2) \cdot \cdots \cdot 2 \cdot 1$ ways to order (i.e. put in a particular order) r items.

Problems

1. A team plays 10 games. How many ways can these games result in:

   3 wins: $C(10,3) = \binom{10}{3} = \frac{10!}{3!\,7!}$

   3 wins and 7 losses: $C(10,3) \cdot C(7,7) = C(10,3) = \binom{10}{3,7}$

   3 wins, 5 losses and 2 ties: $C(10,3) \cdot C(7,5) \cdot C(2,2)$

   $= \frac{10!}{3!\,\cancel{7!}} \cdot \frac{\cancel{7!}}{5!\,\cancel{2!}} \cdot \frac{\cancel{2!}}{2!\,0!} = \frac{10!}{3!\,5!\,2!} = \binom{10}{3,5,2}$

2. How many ways can a group of 100 students be assigned to dorms A, B and C:

   25 to Dorm A 40 to Dorm B 35 to Dorm C.

$C(100,25) \cdot C(75,40) \cdot C(35,35) = \frac{100!}{25!\,\cancel{75!}} \cdot \frac{\cancel{75!}}{40!\,\cancel{35!}} \cdot \frac{\cancel{35!}}{35!\,0!}$

$= \frac{100!}{25!\,40!\,35!} = \binom{100}{25, 40, 35}$

3. Of 14 applicants to a software company, 3 will be hired to work on programming languages, 4 will work on word processing software, and 5 will work on spreadsheet software. In how many ways can the company hire and assign the 12 new employees?

$$\binom{14}{3, 4, 5, 2}$$

not hired

4. How many ways to divide 10 players into two 5-player groups that will play each other?

$$\binom{10}{5,5} \Big/ 2!$$

5. How many ways to divide 10 persons into two 5-player groups: the starting 5 and the 5 who will be subs?

$$\binom{10}{5,5}$$

6. How many ways to divide 20 persons into 4 groups of 5 each?

$$\binom{20}{5,5,5,5} \Big/ 4!$$

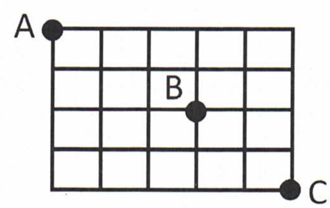
7. How many ways to divide 20 persons into groups of sizes 2, 2, 2 and 14?

$$\binom{20}{2,2,2,14} \Big/ 3!$$

8. How many ways to divide 20 persons into groups of sizes 2, 2, 8 and 8?

$$\binom{20}{2,2,8,8} \Big/ 2!\,2!$$

9. Suppose you can only move south (down) or east (right).



How many ways are there to get from A to B? List them.

SSEEE ESSEE EESSE EEESS
SESEE ESESE EESES
SEESE ESEES
SEEES

$C(5,2)$ or $C(5,3) = 10$

Choose S moves · Choose E

How many ways are there to get from B to C?

$C(4,2)$ or $C(4,2) = 6$

How many ways are there to get from A to C?

$C(9,4)$ or $C(9,5) = 126$

How many ways are there to get from A to C that pass through B?

$10 \cdot 6 = 60$ (which < 126)

Restriction: must go through B

No restriction

10. A bag of 10 apples contains 2 rotten apples and 8 good apples. A shopper selects a sample of 3 apples from the bag.

How many different samples are possible?

$C(10,3) = \cdots = 120$

How many samples contain all good apples?

$C(2,0) \cdot C(8,3) = 56$

How many samples contain at least 1 rotten apple?

$\underbrace{C(2,1) \cdot C(8,2)}_{\text{one rotten two good}} + \underbrace{C(2,2) \cdot C(8,1)}_{\text{two rotten one good}} = 64$

OR: $120 - 56 = 64$

11. In how many ways can 6 married couples sit next to each other if

Anyone can sit next to anyone else: $12!$

Each couple must sit together: $12 \cdot 1 \cdot 10 \cdot 1 \cdot 8 \cdot 1 \cdot 6 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1$

OR: $6! \cdot 2^6$

Men and women must alternate (not necessarily as couples):

Anyone → $12 \cdot 6 \cdot 5 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1$

Same gender

The other gender

12. An urn contains 8 red and 4 white balls. You select 4 of the 12. First note that the total number of ways to choose 4 balls from 12 is: $C(12,4)$

Red balls	White balls	Number of possible samples
4	0	$C(8,4) \cdot C(4,0)$
3	1	$C(8,3) \cdot C(4,1)$
2	2	$C(8,2) \cdot C(4,2)$
1	3	$C(8,1) \cdot C(4,3)$
0	4	$C(8,0) \cdot C(4,4)$

13. In how many ways can a residence director assign six students to four dorm rooms if two rooms are doubles, two rooms are singles, and two of the students cannot be placed together?

If no restrictions: $\binom{6}{1,1,2,2} = 180$

If together: $2 \cdot \binom{4}{1,1,2} = 24$

which double "the two" are in

So $180 - 24 = 156$