Section 2.6 Math 141
Input-Output Analysis

Main ideas

There are many real-world problems which lead to a system of equations $AX = B$. In this section is one application. The problem we will solve is: How much to produce if

Amount produced – Amount consumed = Demand

That is, where X is the amount of each product that we produce, D ("demand") is the amount we want to end up with, and A is the input-out (a.k.a. "consumption") matrix, solve for X in $X - AX = D$:

$$X - AX = D \Rightarrow (I - A)X = D \Rightarrow X = (I - A)^{-1}D.$$

(This is like the problem where if $MX = B$ then we have $X = M^{-1}B$.)

The i-th column of matrix A is the amount *consumed* if *producing* 1 unit of product i. The i-th column of matrix $(I - A)^{-1}$ is the amount *needed to produce* if we want to *end up* with 1 unit of product i.

In class

Let's look at Book Example 1. The input-output matrix (a.k.a. consumption matrix):

Produced

$$\begin{array}{c} \begin{array}{ccc} \text{C} & \text{S} & \text{E} \end{array} \\ \begin{bmatrix} 0 & \mathbf{.15} & .43 \\ .02 & .03 & .20 \\ .01 & .08 & .05 \end{bmatrix} \end{array} \begin{array}{c} \text{C} \\ \text{S} \\ \text{E} \end{array} \quad \textit{Consumed}$$

For example, to produce 1 unit (or \$1 worth) of Steel requires/consumes .15 units (or \$.15) of Coal. Where A is the input-output matrix and X is the amount produced, then AX is the amount consumed in the production process:

$$\begin{bmatrix} 0 & .15 & .43 \\ .02 & .03 & .20 \\ .01 & .08 & .05 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.44 \\ .67 \\ .25 \end{bmatrix}$$

So remaining would be $X - AX = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1.44 \\ .67 \\ .25 \end{bmatrix} = \begin{bmatrix} .56 \\ .33 \\ 2.75 \end{bmatrix}$.

Notice $\begin{bmatrix} 0 & .15 & .43 \\ .02 & .03 & .20 \\ .01 & .08 & .05 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ .02 \\ .01 \end{bmatrix}$ $\begin{bmatrix} 0 & .15 & .43 \\ .02 & .03 & .20 \\ .01 & .08 & .05 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .15 \\ .03 \\ .08 \end{bmatrix}$.

So the amount consumed if we produce 1 unit of Product 1 is simply Column 1 of A.

Same for Products/Columns 2 and 3:

$$\begin{bmatrix} 0 & .15 & .43 \\ .02 & .03 & .20 \\ .01 & .08 & .05 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .15 \\ .03 \\ .08 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & .15 & .43 \\ .02 & .03 & .20 \\ .01 & .08 & .05 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .43 \\ .20 \\ .05 \end{bmatrix}.$$

Suppose we wanted to end up with $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ units of $\begin{bmatrix} Coal \\ Steel \\ Electricity \end{bmatrix}$.

How much extra do we need to produce—knowing that some of it will be consumed in the process—in order to end up with those desired amounts?

Here are a few possibilities (after some guessing):

Produced X	Consumed AX	Remaining $X - AX$	Did we produce the right amount?
$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 1.44 \\ .67 \\ .25 \end{bmatrix}$	$\begin{bmatrix} .56 \\ .33 \\ 2.75 \end{bmatrix}$	Not nearly enough.
$\begin{bmatrix} 3.6 \\ 1.7 \\ 3.3 \end{bmatrix}$	$\begin{bmatrix} 1.674 \\ .783 \\ .337 \end{bmatrix}$	$\begin{bmatrix} 1.926 \\ .917 \\ 2.963 \end{bmatrix}$	Not quite enough.
$\begin{bmatrix} 3.8 \\ 1.9 \\ 3.4 \end{bmatrix}$	$\begin{bmatrix} 1.747 \\ .813 \\ .360 \end{bmatrix}$	$\begin{bmatrix} 2.053 \\ 1.087 \\ 3.040 \end{bmatrix}$	Slightly too much.

So we need to produce somewhere between $\begin{bmatrix} 3.6 \\ 1.7 \\ 3.3 \end{bmatrix}$ and $\begin{bmatrix} 3.8 \\ 1.9 \\ 3.4 \end{bmatrix}$. But we can do better than guess. We usually "do some math" to set up the problem and then "do some math" to solve the problem. (Remember the coins example from the first day of class?) See the front of this handout for how we set up and solve this problem.

Let's work the above example where $D = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$,

and then again with $D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Let's look at the 2 x 2 Input-Output Example online.

Notice the spreadsheet online for doing the 3 x 3 version of this problem. If time, let's experiment a bit with some of the numbers.