Math 141 Spring 2024 **Midterm Exam 1** **January 26, 2024**

Name: Solutions

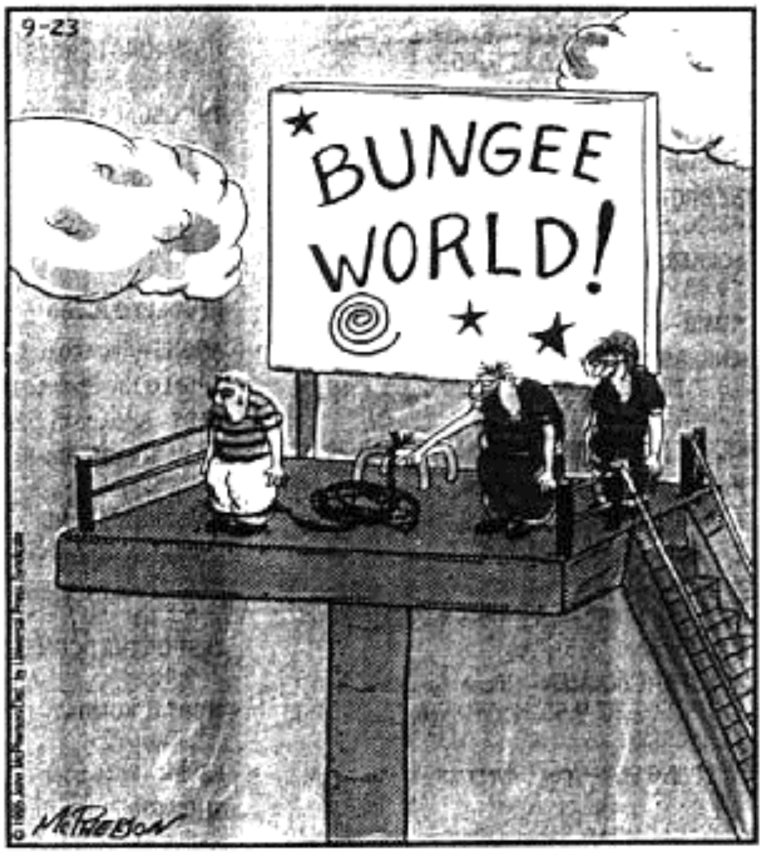
Problem	1	2 / 3	4	5 / 6	7	Total
Possible	20	23	15	24	18	100
Received						

DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 x 5 card
(both sides) of notes,
but no calculator.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.

Close To Home John McPherson



"Okee-doke! Let's just double-check. We're 130 feet up and we've got 45 yards of bungee cord, that's uh ... 90 feet. Allow for 30 feet of stretching, that gives us a total of ...120 feet. Perfect!"

20 points **1.** Answer each of the following questions. No explanation is needed.

[✓ **T**] **F** A system of 3 equations and 3 unknowns could possibly have no solution.

[✓ **T**] **F** A system of 3 equations and 3 unknowns could possibly have a unique solution.

[✓ **T**] **F** A system of 3 equations and 3 unknowns could possibly have infinite solutions.

[✓ **T**] **F** A system of 2 equations and 4 unknowns could possibly have no solution.

T [✓ **F**] A system of 2 equations and 4 unknowns could possibly have a unique solution.

[✓ **T**] **F** A system of 2 equations and 4 unknowns could possibly have infinite solutions.

[✓ **T**] **F** Matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is its own inverse.

T [✓ **F**] $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 21 \\ 12 & 32 \end{bmatrix}$.

T [✓ **F**] $\begin{aligned} 2x + 3y &= a \\ 4x + 5y &= b \end{aligned}$ might or might not have a solution, depending on the values of a and b.

It for sure has a solution

[✓ **T**] **F** It is possible to choose values for a and b so that $\begin{bmatrix} 2 & 7 \\ a & b \end{bmatrix}$ has an inverse.

14 points **2.** Suppose I have some nickels (5 cents each) and dimes (10 cents each). I have 13 coins total, 90 cents total, and I have 3 more nickels than dimes (so $n = d + 3$). How many of each type of coin do I have? Solve this by coming up with the three equations that correspond to these three conditions (13 coins total, 90 cents total, and 3 more nickels than dimes), then doing Gauss-Jordan Elimination to find the solution(s) to this system of equations. <u>Don't just guess the solution</u>. Or show that there is no solution, if that is the case.

$n + d = 13$

$5n + 10d = 90$

$n - d = 3$

$$\left[\begin{array}{cc|c} 1 & 1 & 13 \\ 5 & 10 & 90 \\ 1 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 13 \\ 0 & 5 & 25 \\ 0 & -2 & -10 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 13 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

So $n = 8$, $d = 5$.

9 points **3.** Rework the previous problem, but with the modified restriction that we have 100 cents, rather than 90 cents (but with the other conditions remaining the same).

$$\left[\begin{array}{cc|c} 1 & 1 & 13 \\ 5 & 10 & 100 \\ 1 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 13 \\ 0 & 5 & 35 \\ 0 & -2 & -10 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 13 \\ 0 & 1 & 7 \\ 0 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & -2 \end{array}\right]$$

No solution.

15 points **4.** Solve for x, y and z in

$$\begin{aligned} x + y + z &= 1 \\ 2x + y - z &= 0 \\ x + y + 2z &= 1 \end{aligned}$$

by <u>finding the inverse</u> of the coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

and using it to find the values of x, y and z. Use the Gauss-Jordan Method for finding the inverse. You should not encounter any fractions in finding it. <u>Show work</u>. <u>Don't just guess answers</u>.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 1 & 2 \\ 0 & 1 & 0 & 5 & -1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} -3 & 1 & 2 \\ 5 & -1 & -3 \\ -1 & 0 & 1 \end{matrix}}_{A^{-1}}$$

So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 \\ 5 & -1 & -3 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

10 points **5.** We are interested in solving the following system of equations,

$$3x + 2y = 7$$
$$6x + ay = b$$

where a and b are some constants whose values have not yet been decided. Give an example of values of a and b that result in the system having:

No solution: $a = 4$ $b \neq 14$ parallel lines

One solution: $a \neq 4$ $b =$ any number

Infinite solutions: $a = 4$ $b = 14$ same line

14 points **6.** Find the solution(s) to each of the following linear systems. If a system has more than one solution, give the general solution and then give *at least two* specific solutions. If a system has no solution, state that.

$$2x + 4y = 7$$
$$-x - y = -2$$

$$\left[\begin{array}{cc|c} 2 & 4 & 7 \\ -1 & -1 & -2 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$

OR $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$

$$x + y - z + 2w = 5$$
$$-x - y + 3z = 7$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 5 \\ -1 & -1 & 3 & 0 & 7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 5 \\ 0 & 0 & 2 & 2 & 12 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 5 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 11 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right]$$

$x + y + 3w = 5$
$z + w = 6$

$x = -y - 3w + 5$
$y =$ free
$z = -w + 6$
$w =$ free

General solution

e.g. $\begin{bmatrix} 5 \\ 0 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ 5 \\ 1 \end{bmatrix}$, or ...

Specific solutions

18 points **7.** A company produces two items, but uses up some of each product in the production process, as described by the input-output (consumption) matrix

$$A = \begin{bmatrix} .6 & .2 \\ 0 & .5 \end{bmatrix}.$$

Note for this problem that $(.5)(.4) = .2$, and that $\frac{.5}{.2} = \frac{5}{2}$ and $\frac{.4}{.2} = 2$.

2 points How much of each product would be *consumed* if you *produced* 10 units of each product?

$$\begin{bmatrix} .6 & .2 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

2 points How much of each product is *remaining* if you *produced* 10 units of each product?

$$\begin{bmatrix} 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

2 points How much *more* of each product would be *consumed* if you produced *one more unit* of product 1?

$$\begin{bmatrix} .6 & .2 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} 11 \\ 10 \end{bmatrix} = \begin{bmatrix} 8.6 \\ 5 \end{bmatrix}, \text{ so } \begin{bmatrix} .6 \\ 0 \end{bmatrix} \leftarrow \text{Column 1 of } A$$

12 points How much would you need to produce in order to *end up* with 10 units of each product? (Use the formula for finding the 2 × 2 matrix in this problem.) What is one thing about your solution that makes you think it is reasonable, i.e. that it could be the correct answer?

$$(I-A)^{-1} = \begin{bmatrix} .4 & -.2 \\ 0 & .5 \end{bmatrix}^{-1} = \frac{1}{.2}\begin{bmatrix} .5 & .2 \\ 0 & .4 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 1 \\ 0 & 2 \end{bmatrix}$$

$$\nearrow$$

$$(.4)(.5) - (-.2)(0)$$

So produce $\begin{bmatrix} \frac{5}{2} & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 35 \\ 20 \end{bmatrix}$,

which $> \begin{bmatrix} 10 \\ 10 \end{bmatrix}$