Math 141 Spring 2023 **Midterm Exam 2** **February 20, 2023**

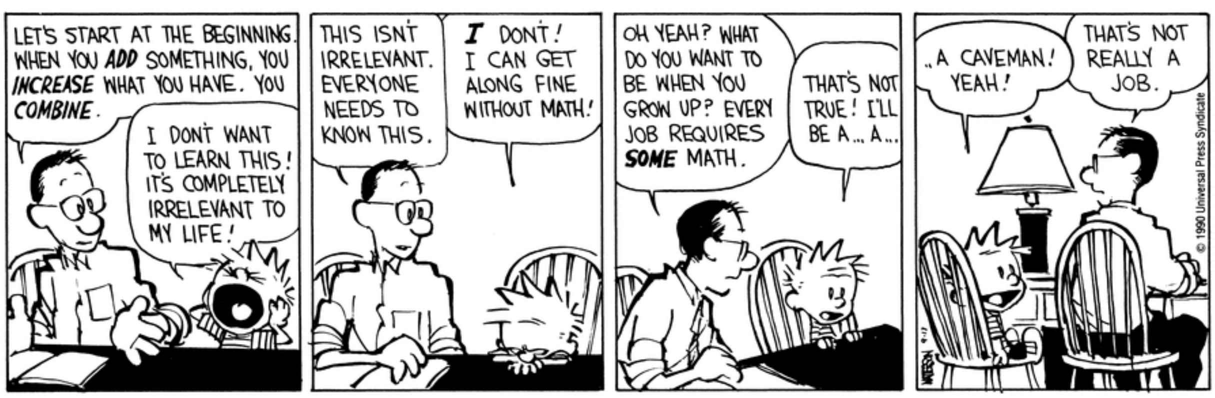
Name: Solutions

Problem	1	2 / 3	4 / 5	6 / 7	8	9	Total
Possible	8	16	14	22	20	20	100
Received							

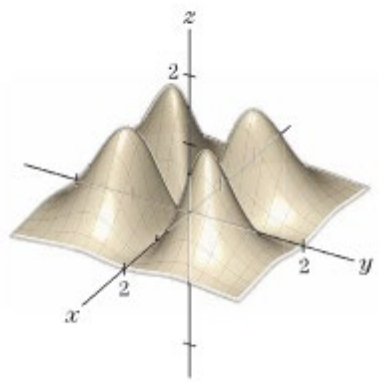
DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 × 5 card of handwritten notes and a calculator.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.

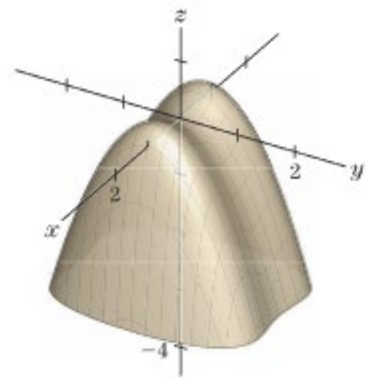


8 points 1. Match the graphs of the following four functions to the level curves show below the functions. Just write a letter (**a** or **b** or **c** or **d**) next to each.



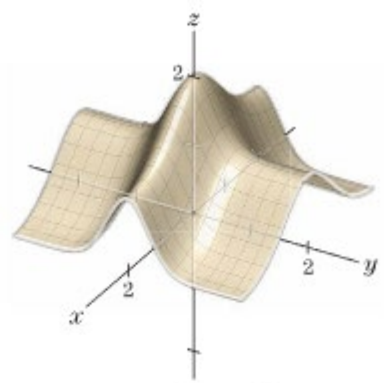
$$z = \frac{15x^2y^2e^{-x^2-y^2}}{x^2+y^2}$$

$\boxed{C}$



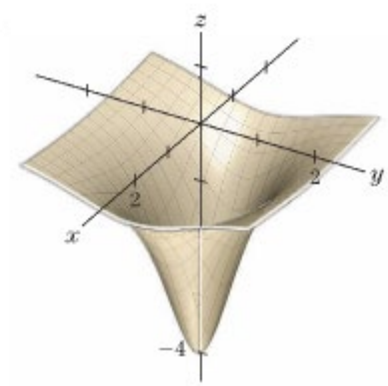
$$z = x^2 - x^4 - y^2$$

$\boxed{B}$



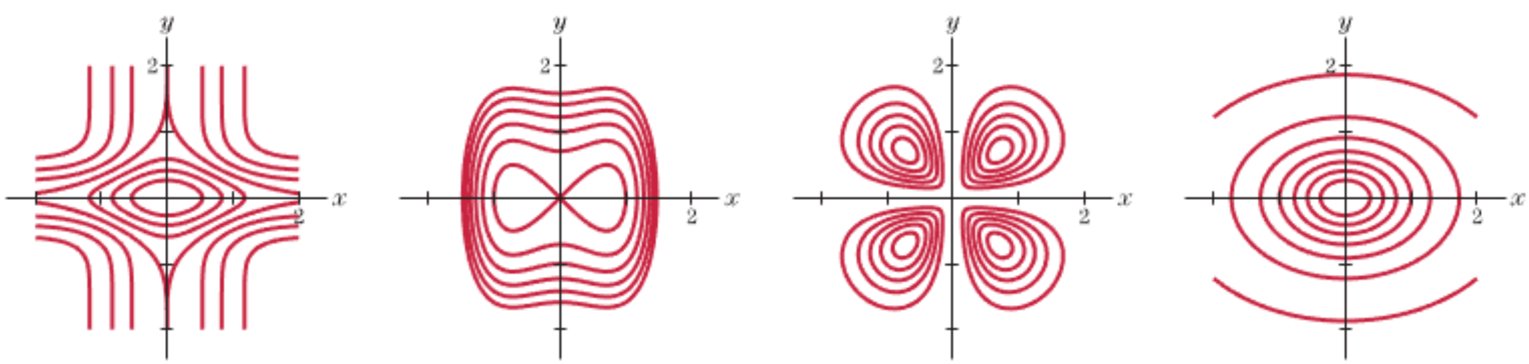
$$z = e^{-x^2} + e^{-4y^2}$$

$\boxed{A}$



$$z = \frac{-4}{1+x^2+2y^2}$$

$\boxed{D}$



(a) (b) (c) (d)

12 points 2. Consider the production function $f(x,y) = 32x^{1/4}y^{3/4}$, which gives the number of units of goods produced when x units of labor and y units of capital are used.

/4 Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = 32 \cdot \frac{1}{4} x^{-3/4} y^{3/4} = \frac{8y^{3/4}}{x^{3/4}}$$

$$\frac{\partial f}{\partial y} = 32x^{1/4} \cdot \frac{3}{4} y^{-1/4} = \frac{24x^{1/4}}{y^{1/4}}$$

/4 Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x = 81$ and $y = 16$. Note that $81^{1/4} = 3$ and $16^{1/4} = 2$.

$$\frac{\partial f}{\partial x}(81,16) = \frac{8 \cdot 16^{3/4}}{81^{3/4}} = \frac{8 \cdot 2^3}{3^3} = \frac{64}{27}$$

$$\frac{\partial f}{\partial y}(81,16) = \frac{24 \cdot 81^{1/4}}{16^{1/4}} = \frac{24 \cdot 3}{2} = 36$$

/2 Find the marginal productivity of capital of f at $x = 81$ and $y = 16$.

/2 Using above results, approximately what is $f(81,17) - f(81,16)$?

This is approximately how much f increases if y increases by 1.

4 points 3. Suppose that $f(10,10) = 100$, $\frac{\partial f}{\partial x}(10,10) = 4$ and $\frac{\partial f}{\partial y}(10,10) = 3$. Estimate $f(11.5, 9)$. Show pertinent details/work.

$$\approx 100 + 1.5(4) - 1(3) = 103$$

4 points 4. Suppose the distance D that a car can travel depends on the amount of gas g in the car and the total weight w of the passengers in the car. Circle > 0 or $= 0$ or < 0 for the following derivatives of D.

/2 Should $\frac{\partial D}{\partial g}$ be > 0 or $= 0$ or < 0? If $g \uparrow$, then $D \uparrow$

/2 Should $\frac{\partial D}{\partial w}$ be > 0 or $= 0$ or < 0? If $w \uparrow$, then $D \downarrow$

10 points 5. Find the point(s) at which $f(x,y) = 5x^2 - 2xy + 2y^2 - 6y + 7$ has minimum(s) and maximum(s), and determine what type of point (min or max or neither) each point is.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 10x - 2y = 0 \\ \frac{\partial f}{\partial y} &= -2x + 4y - 6 = 0 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x &= \frac{1}{3} \\ y &= \frac{5}{3} \end{aligned}}$$

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}$$

$$= 10 \cdot 4 - (-2)^2$$

$= 36$ for any x, y.

$D\left(\frac{1}{3}, \frac{5}{3}\right) = 36 > 0 \Rightarrow$ min or max

$\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{3}, \frac{5}{3}\right) = 10 > 0 \Rightarrow \boxed{\text{min}}$!

14 points 6. Find the following derivatives.

For $f(x,y) = e^{xy^2}$

/3 $\frac{\partial f}{\partial x} = e^{xy^2} \cdot \frac{\partial}{\partial x}(xy^2) = e^{xy^2} \cdot y^2$

/4 $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[e^{xy^2} \cdot y^2\right] = e^{xy^2} \cdot x \cdot 2y \cdot y^2 + e^{xy^2} \cdot 2y$

using product rule.

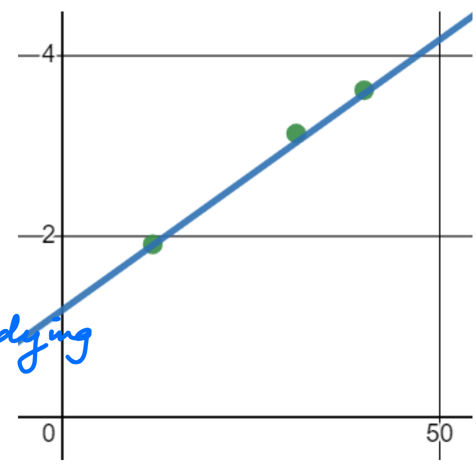
For $g(x,y) = x^2 \cdot \ln(y^3)$

/3 $\frac{\partial g}{\partial y} = x^2 \cdot \frac{1}{y^3} \cdot \frac{\partial}{\partial y} y^3 = x^2 \cdot \frac{1}{y^3} \cdot 3y^2 = \frac{3x^2}{y}$

/4 $\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left[3x^2 \cdot y^{-1}\right] = 3x^2 \cdot (-1) y^{-2} = -\frac{3x^2}{y^2}$

8 points 7. Suppose that I asked three students how many hours they study per week and what their current GPA is, and found a least squares line based on their responses of

$$GPA \approx 1.18 + .06 * hours\ studied$$



/4 What do the values of 1.18 and .06 tell us?

If hrs = 0, then GPA = 1.18: GPA if no studying

If hrs ↑ 1, then GPA ↑ .06: increase in GPA for each additional hour studied.

/2 What GPA would result from studying 10 hours per week?

$1.18 + .06(10) = 1.78$

/2 How many hours per week would you need to study to get a GPA of 3.58?

$3.58 = 1.18 + .06 * hrs$

$\Rightarrow hrs = \frac{3.58 - 1.18}{.06} = 40.$

20 points 8. Suppose we want a very tiny home with dimensions x, y and z to have volume 1 cubic yard, so $xyz = 1$. Suppose that the daily loss (through the walls, ceiling and floor) of heat is given by

$$H = xy + 2xz + 4yz.$$

Find the dimensions of the home which minimize heat loss H.

For this problem, find the solution <u>by substituting</u> $z = \frac{1}{xy}$ <u>into</u> $H = xy + 2xz + 4yz$ and (1) find the values of x and y at which H is minimized, and (2) show that at these x and y values function H is minimized (use second derivatives and $D(x, y)$).

$$H = xy + 2x\left(\frac{1}{xy}\right) + 4y\left(\frac{1}{xy}\right)$$

$$= xy + 2y^{-1} + 4x^{-1}$$

$$\frac{\partial H}{\partial x} = y - 4x^{-2} = y - \frac{4}{x^2} = 0 \Rightarrow y = \frac{4}{x^2}$$

$$\frac{\partial H}{\partial y} = x - 2y^{-2} = x - \frac{2}{y^2} = 0 \Rightarrow x = \frac{2}{\left(\frac{4}{x^2}\right)^2} = \frac{2}{16} \cdot x^4$$

$$\Rightarrow x^4 = 8x \Rightarrow x^3 = 8 \Rightarrow \boxed{x = 2}$$

$$\Rightarrow \boxed{y = \frac{4}{2^2} = 1.}$$

$$D(x,y) = \frac{\partial^2 H}{\partial x^2} \cdot \frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 = (8x^{-3})(4y^{-3}) - 1^2 = \frac{32}{x^3 y^3} - 1$$

$$D(2,1) = \frac{32}{2^3 \cdot 1^3} - 1^2 = 3 > 0 \Rightarrow \text{min or max}$$

$$\frac{\partial^2 H}{\partial x^2}(2,1) = \frac{8}{2^3} > 0 \Rightarrow \text{min.}$$

And $\boxed{z = \frac{1}{2 \cdot 1} = \frac{1}{2}}$

20 points 9. Same as previous problem: minimize

$$H = xy + 2xz + 4yz$$

with the constraint that $xyz = 1$. But now solve this problem by <u>using the Lagrange Multiplier Method</u>.

$$F(x,y,z,\lambda) = xy + 2xz + 4yz + \lambda(1 - xyz)$$

$$\frac{\partial F}{\partial x} = y + 2z - \lambda yz = 0 \Rightarrow \lambda = \frac{y+2z}{yz} = \frac{1}{z} + \frac{2}{y} \quad (1)$$

$$\frac{\partial F}{\partial y} = x + 4z - \lambda xz = 0 \Rightarrow \lambda = \frac{x+4z}{xz} = \frac{1}{z} + \frac{4}{x} \quad (2)$$

$$\frac{\partial F}{\partial z} = 2x + 4y - \lambda xy = 0 \Rightarrow \lambda = \frac{2x+4y}{xy} = \frac{2}{y} + \frac{4}{x} \quad (3)$$

$$(1),(2) \Rightarrow \frac{2}{y} = \frac{4}{x} \Rightarrow y = \frac{x}{2}$$

$$(1),(3) \Rightarrow \frac{1}{z} = \frac{4}{x} \Rightarrow z = \frac{x}{4}$$

In constraint:

$$x \cdot \left(\frac{x}{2}\right) \cdot \left(\frac{x}{4}\right) = 1 \Rightarrow x^3 = 8$$

$$x = 2$$

$$\Rightarrow y = \frac{2}{2} = 1$$

$$\Rightarrow z = \frac{2}{4} = \frac{1}{2}$$

(as found in previous problem)