Math 141 Fall 2023 **Midterm Exam 1** **September 8, 2023**

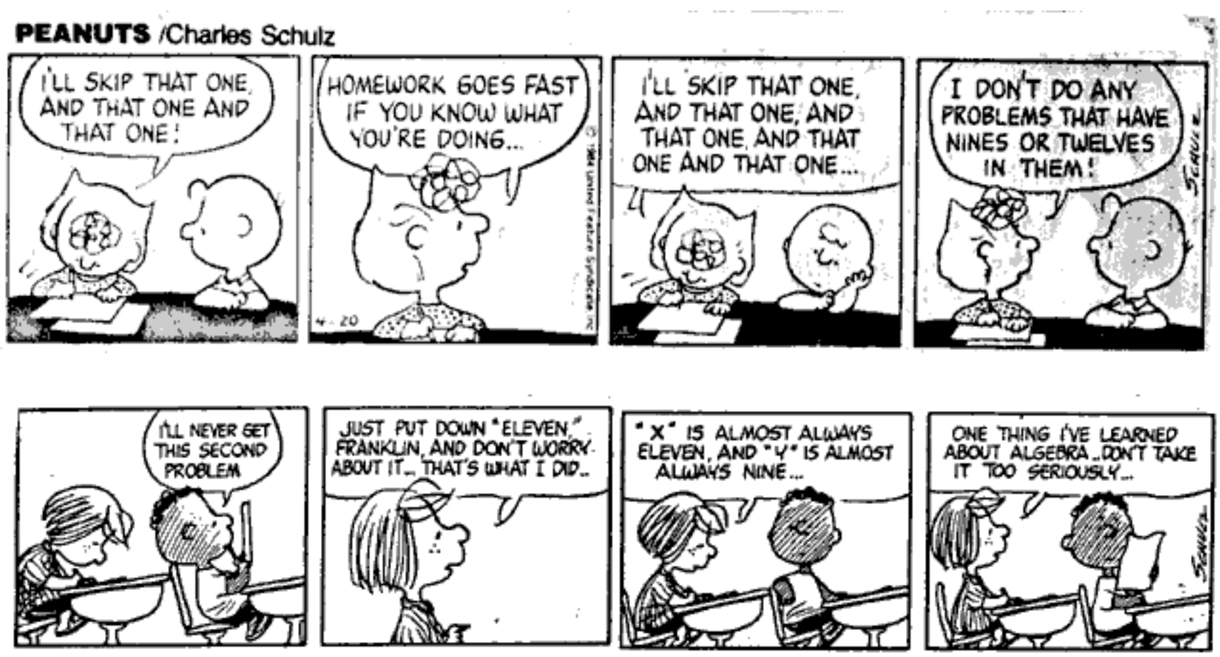
Name: Solutions

Problem	1	2	3	4	5	6/EC	Total
Possible	10	20	18	23	20	9	100
Received							

DO NOT OPEN YOUR EXAM UNTIL TOLD TO DO SO.

You may use a 3 × 5 card of handwritten notes and a calculator.

FOR FULL CREDIT, SHOW ALL WORK RELATED TO FINDING EACH SOLUTION.



10 points 1. Answer each of the following True/False questions. No explanation is needed.

T **F** The system of equations $\begin{aligned} 2x - 3y + 5z &= 2 \\ 2x - 3y + 4z &= a \end{aligned}$ may or may not have an infinite number of solutions, depending on what a is.

It is 100% certain that there will be infinite solutions, no matter what a is.

T **F** The system of equations $\begin{aligned} 2x - 3y + 5z &= 2 \\ 2x - 3y + 5z &= a \end{aligned}$ may have exactly one solution, depending on what a is.

If $a = 2$, there are infinite solns.
If $a \neq 2$, there is no solution.

T **F** The matrix $\begin{bmatrix} 1 & 2 \\ 3 & a \end{bmatrix}$ has an inverse no matter what a is.

Only if $a \neq 6$.

T **F** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ does not have an inverse, then $AX = B$ will not have a solution.

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has infinite solns.

T **F** For $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, the system $AX = B$ may not have a solution, depending on what B is.

It will for sure has the solution of $X = A^{-1}B$.

It was purely unintentional that all five were false.

$b \quad f \quad d$

20 points 2. A group will order some burgers, fries and drinks. Each burger costs \$10, each order of fries costs \$6, and each drink costs \$2. Some (or all) of the following conditions will be met:

1. They will order 9 items. $b + f + d = 9$
2. They will spend a total of \$50. $10b + 6f + 2d = 50$
3. They will order one more burger than drinks. $b = d + 1 \Rightarrow b - d = 1.$

In each of the following three problems:

- If there is one solution, find it.
- If there is no solution, state so and explain why (show some matrix work).
- If there are infinite solutions, find the general solution and find two particular solutions.

/10 How many of each item will they buy if they need to meet conditions 1 and 2?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 10 & 6 & 2 & 50 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -4 & -8 & -40 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 10 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 10 \end{array}\right] \text{ so } \begin{array}{l} b = d - 1 \\ f = -2d + 10 \\ d = \text{free} \end{array}$$

e.g. $\begin{matrix} 0 \\ 8 \\ 1 \end{matrix}$, $\begin{matrix} 1 \\ 6 \\ 2 \end{matrix}$, etc.

All values should be ≥ 0.

/10 How many of each item will they buy if they need to meet conditions 1, 2 and 3?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 10 & 6 & 2 & 50 \\ 1 & 0 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -4 & -8 & -40 \\ 0 & -1 & -2 & -8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 10 \\ 0 & 1 & 2 & 8 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

No solution.

18 points 3. Suppose a certain economy consists of two sectors. Suppose that this economy has the input-output (consumption) matrix $A = \begin{bmatrix} .2 & .3 \\ .4 & .2 \end{bmatrix}$.

/2 How much of each product would be *consumed* if you *produced* 100 units of each product?

$$\begin{bmatrix} .2 & .3 \\ .4 & .2 \end{bmatrix}\begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 50 \\ 60 \end{bmatrix}$$

/2 How much of each product is *remaining* if you *produced* 100 units of each product?

$$\begin{bmatrix} 100 \\ 100 \end{bmatrix} - \begin{bmatrix} 50 \\ 60 \end{bmatrix} = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

/2 How much *more* of each product would be *consumed* if you produced *one more unit* of product 2?

$$\begin{bmatrix} .2 & .3 \\ .4 & .2 \end{bmatrix}\begin{bmatrix} 100 \\ 101 \end{bmatrix} = \begin{bmatrix} 50.3 \\ 40.2 \end{bmatrix}, \text{ so } \begin{bmatrix} .3 \\ .2 \end{bmatrix} \text{ more, Col. 2 of } A.$$

/10 How much would you need to produce in order to *end up* with 100 units of each product? (Use the formula for finding the 2×2 matrix in this problem.) What is one thing about your solution that makes you think it is reasonable, i.e. that it could be the correct answer?

$$(I-A)^{-1} = \begin{bmatrix} .8 & -.3 \\ -.4 & .8 \end{bmatrix}^{-1} = \frac{1}{\underbrace{(.8)(.8)-(-.3)(-.4)}_{.52}} \begin{bmatrix} .8 & .3 \\ .4 & .8 \end{bmatrix}$$

$$\approx \underbrace{\begin{bmatrix} 1.54 & .58 \\ .77 & 1.54 \end{bmatrix}}_{(I-A)^{-1}} \underbrace{\begin{bmatrix} 100 \\ 100 \end{bmatrix}}_{D} = \begin{bmatrix} 212 \\ 231 \end{bmatrix}$$

/2 How much *more* of each product would be needed if you wanted to *end up* with *one more* unit of product 2 (so 100 units of product 1, and 101 units of product 2).

Col. 2 of $(I-A)^{-1}$: $\begin{bmatrix} .58 \\ 1.54 \end{bmatrix}$.

23 points 4. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

/10 Use Gauss-Jordan elimination to find the inverse of A.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

A^{-1}

/4 Use A^{-1} to find the solution to $\begin{aligned} x + \ \ y + \ \ z &= 2 \\ x + 2y + 3z &= 2 \\ x + \ \ y + 2z &= 3 \end{aligned}$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

/9 Now use Gauss-Jordan elimination to find the solution to $\begin{aligned} x + \ \ y + \ \ z &= 2 \\ x + 2y + 3z &= 2 \\ x + \ \ y + 2z &= 3 \end{aligned}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Same as above

20 points 5. Find the solutions to each of the following linear systems. If a system has more than one solution, <u>give the general solution and then give two specific solutions</u>. If a system has no solution, state that. <u>Show work—don't just write answers</u>.

/8

$$\begin{aligned} x + y - 2z + 3w &= 5 \\ -2x - 2y + 2z - 2w &= 6 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 5 \\ -2 & -2 & 2 & -2 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 5 \\ 0 & 0 & -2 & 4 & 16 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -11 \\ 0 & 0 & 1 & -2 & -8 \end{array}\right] \quad \begin{aligned} x + y \quad - w &= -11 \\ z - 2w &= -8 \end{aligned}$$

so
$$\begin{aligned} x &= -y + w - 11 \\ y &= \text{free} \\ z &= 2w - 8 \\ w &= \text{free} \end{aligned} \quad \text{, e.g. } \begin{matrix} -11 \\ 0 \\ -8 \\ 0 \end{matrix} \ , \ \begin{matrix} -11 \\ 1 \\ -6 \\ 1 \end{matrix} \ , \ldots$$

/6

$$\begin{aligned} 2x - 6y &= 10 \\ -4x + 12y &= -20 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & -6 & 10 \\ -4 & 12 & -20 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x &= 3y + 5 \\ y &= \text{free} \end{aligned} \quad \text{, e.g. } \begin{matrix} 5 \\ 0 \end{matrix} \ , \ \begin{matrix} 8 \\ 1 \end{matrix} \ , \ldots$$

/6

$$\begin{aligned} n + d &= 12 \\ n - 5d &= 0 \\ 5n + 10d &= 70 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 1 & -5 & 0 \\ 5 & 10 & 70 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & -6 & -12 \\ 0 & 5 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 1 & 2 \\ 0 & 5 & 10 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \begin{aligned} n &= 10 \\ d &= 2 \end{aligned}$$

9 points 6. Each of the following is the final matrix of a Gauss-Jordan elimination process. Give the solutions to the corresponding systems of linear equations. You can use x and y (and z, if needed) for the unknowns.

Final matrix	Solution
$\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$	No solution
$\left[\begin{array}{ccc\|c} 1 & -1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$	$x - y = 6 \Rightarrow x = y + 6$, $y = $ free, $z = 0$
$\left[\begin{array}{cc\|c} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$	$x = 6$, $y = 3$ (There is no z.)

2 points EC. Extra credit. Be sure to show your work.

If $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $A^5 = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}$, what is A?

$$A^2 \cdot A^2 \cdot A = A^5,$$

$$\text{so } A = (A^2)^{-1} \cdot (A^2)^{-1} \cdot A^5$$

$$(A^2)^{-1} = \frac{1}{1 \cdot 2 - 1 \cdot 1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\text{so } A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$