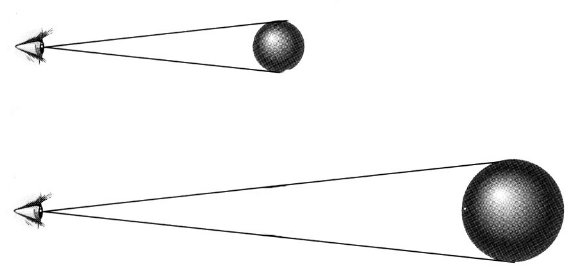
The Small Angle Formula

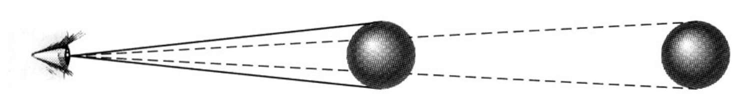
We discovered in lab that you can use angular distance to estimate the relative position of two celestial objects. Remember, though, that angular distance is not the same as actual distance.

Take, for example, two celestial objects that are at two different distances with the larger body farther away:



In this example, the larger object appears to have the identical angular size as the smaller body.

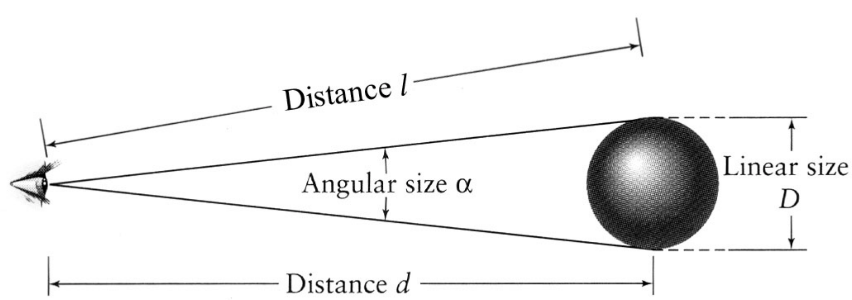
In a second example, the two celestial bodies are the same physical size but one is farther away than the other.



In this case the farther object appears smaller to the observer but is reality the same size as the closer.

Despite this seemingly irresolvable conflict, we *can* convert angular size to actual size *if* we know the distance to the body. Conversely, if we know the actual size and can measure the angular size, it is possible to determine the distance to the object.

First, we establish the geometrical picture of the relationship of actual size, angular size, and linear distance the object is away from the observer:



The trigonometric relationship of the linear size of the of the object, D, the distance to the edge of the object, l, and the apparent angle subtended by the object, α, is

$$D = l\sqrt{2 - 2\cos\alpha}$$

Without trigonometric proof, if the distance, *d*, is so large that the length of either long leg, *l*, of the triangle is almost (but not exactly) equal to *d*, then the trigonometric relationship simplifies to the "small-angle formula",

$$D = \frac{2\pi}{360^\circ}\alpha d$$

if the angle subtended by the object is measured in degrees. The distances to celestial objects is so large that their apparent angular sizes are so small that they often must be measured in arc minutes (1 arc min = 1' = 1/60 degree) or arc seconds (1 arc sec = 1" = 1/3600 degree). If the angle, a, is measured in arc sec, the small-angle formula becomes

$$D = \frac{2\pi}{3600 \times 360^\circ}\alpha d = \frac{2\pi}{1.296 \times 10^6}\alpha d$$

Or just simply

$$D = \frac{\alpha d}{206,265}$$

This is one of several valuable equations to have available for analyzing astronomical measurements. Whatever distance units are used for *D* must also be used for *d*.

Example 1:

On January 1, 1998, the planet Uranus was at a distance of 20.721 AU from Earth (recall that 1 AU = 1.496 x 10^8 km = 93 million miles). Using an eyepiece micrometer, an astronomer accurately measured the angular diameter of the planet to be 3.40". What is the diameter of Uranus in kilometers?

Solution:

Write the important equation for solving the problem. Since the angular diameter is given in arc seconds, the form of the small-angle formula used is:

$$D = \frac{\alpha d}{206,265}$$

Since it is the diameter of the planet that we want, the equation can be left exactly as it is written.
The distance to Uranus is in AU but we need it in kilometers to make *D* come out in kilometers.

$$d = 20.721 \text{ AU} \times 1.496 \times 10^8 \, \tfrac{\text{km}}{\text{AU}} = 3.100 \times 10^9 \text{ km}$$

Now it's just a matter of filling in all of the known variables and calculating:

$$D = \frac{3.40 \times 3.100 \times 10^9 \text{ km}}{206,265} = 51099 \text{ km}$$

Using *Voyager* data, the diameter of Uranus is 51,118 km. The Earth-based measurement agrees very well.

Example 2:

A telescope with a 30-cm (12”) aperture can theoretically resolve angles to 0.5 arcsec when the “seeing” conditions are excellent. What is the diameter of the smallest crater which can been seen through this telescope? The average distance to the Moon is 384,000 km (about 239,000 mi).

Solution:

This problem is really no different than the last. Substitute the values for each of the variables

$$D = \frac{\alpha d}{206,265} = \frac{0.5 \times 384,000 \text{ km}}{206,265} = 0.93 \text{ km}$$

Considering that the landing modules used by the astronauts in the Apollo missions of the late 1960’s and 1970’s were only several meters across, there’s not a chance that a small earth-based telescope could ever see spacecraft that in which they landed.