Celestial Mechanics
Solutions

Example 2:
What is the distance from the surface of the Earth of a geosynchronous satellite? A geosynchronous satellite has a period that exactly equals the Earth's sidereal day (23 h 56 min 4 s = 23.93 h).

Use Newton's modification of Kepler's 3rd law. Since the satellite has a miniscule mass compared to the Earth, the combined mass of the Earth and the satellite is just the mass of the Earth. The period of revolution is 23.93 h.

$m = 5.974 \times 10^{24}$ kg
$P = 23.93$ h x 3600 s/h = 86,148 s
$G = 6.6726 \times 10^{-11}$ N·m^2/kg^2
Equatorial diameter of Earth = 12,756 km
Equatorial radius of Earth = $\dfrac{12756 \text{ km}}{2} \times 1000 \frac{\text{m}}{\text{km}} = 6.378 \times 10^6 \text{ m}$

$$P^2 = \left(\frac{4\pi^2}{G(m_1 + m_2)} \right) a^3$$

$$86148^2 = \left(\frac{4\pi^2}{6.6726 \times 10^{-11} \frac{\text{N} \times \text{m}^2}{\text{kg}^2} (5.974 \times 10^{24} \text{ kg})} \right) a^3$$

$$a = \sqrt[3]{7.4936 \times 10^{22}} = 4.216 \times 10^7 \text{ m}$$

But this is the distance from the center of the Earth. Subtract the distance from the center of the Earth to the surface:

$d = 4.216 \times 10^7$ m – 6.378×10^6 m = 3.578×10^7 m

in kilometers: $\boxed{d = 35{,}780 \text{ km}}$

(The answer differs from the handout only because of differences in rounding intermediate answers.)

Example 3:
Accurately determine the mass of the Sun using only the following planetary and planetary satellite data.

Jupiter

Synodic period:	398.9 days
Average Distance from Sun	7.783×10^8 km

Callisto (outermost large moon of Jupiter)

Average distance from center of Jupiter:	1,883,000 km
Orbital period (sidereal):	16.689 days

First get m_{Jupiter} from orbital elements of Callisto, then get $m_{\odot}$ from orbital elements of Jupiter.

1. P_{Callisto} = 16.689 d x 24 $^{\text{h}}/_{\text{d}}$ x 3600 $^{\text{s}}/_{\text{h}}$ = 1.4419 x 10^6 s
   a_{Callisto} = 1.883 x 10^6 km = 1.883 x 10^9 m.

$$P^2 = \left(\frac{4\pi^2}{G(m_1+m_2)}\right)a^3$$

But $m_{\text{Callisto}} \ll m_{\text{Jupiter}}$, so...

$$P^2 = \left(\frac{4\pi^2}{Gm_{\text{Jupiter}}}\right)a^3$$

$$m_{\text{Jupiter}} = \left(\frac{4\pi^2}{GP^2}\right)a^3 = \left(\frac{4\pi^2}{(6.6726\times10^{-11}\ \text{Nm}^2/\text{kg}^2)(1.4419\times10^6\text{s})}\right)(1.883\times10^9\text{m})^3$$

$$\boxed{m_{\text{Jupiter}} = 1.90\times10^{27}\,\text{kg}}$$

2. First, get sidereal period of Jupiter…

$$\frac{1}{P} = \frac{1}{E} - \frac{1}{S} = \frac{1}{365.26\text{d}} - \frac{1}{398.9\text{d}} = 2.3088\times10^{-4}\,\text{d}^{-1}$$

$$P_{\text{Jupiter}} = 4331.2\text{d} = 3.742\times10^8\,\text{s}$$

Use Newton's modification of Kepler's 3rd law again...

$$P^2 = \left(\frac{4\pi^2}{G(m_\odot + m_{\text{Jupiter}})}\right)a^3$$

$$(m_\odot + m_{\text{Jupiter}}) = \left(\frac{4\pi^2}{GP^2}\right)a^3 = \left(\frac{4\pi^2}{(6.6726\times10^{-11}\ \text{Nm}^2/\text{kg}^2)(3.742\times10^8\text{s})}\right)(7.783\times10^{11}\text{m})^3$$

$$(m_\odot + m_{\text{Jupiter}}) = 1.992\times10^{30}\,\text{kg}$$

$$m_\odot = 1.992\times10^{30}\,\text{kg} - 1.90\times10^{27}\,\text{kg} = \boxed{1.990\times10^{30}\,\text{kg}}$$

Example 4:
What is the combined mass of the binary star system in which the distance between the primary star and its orbiting companion is 10 AU (1.5 x 10^9 km) and the period is 5 y?

Solution:
8.0 x 10^{31} kg

This is one of the most important applications of Kepler's 3$^{\text{rd}}$ law; *i.e.*, measuring the mass of stars.

$$a = 1.5\times10^{12}\,\text{m}$$

$$P = 5\text{y}\times\tfrac{365.26\text{d}}{\text{y}}\times24\tfrac{\text{h}}{\text{d}}\times3600\tfrac{\text{s}}{\text{h}} = 1.578\times10^8\,\text{s}$$

$$P^2 = \left(\frac{4\pi^2}{G(m_1+m_2)}\right)a^3 = \left(\frac{4\pi^2}{Gm_{\text{Total}}}\right)a^3$$

$$m_{\text{Total}} = \left(\frac{4\pi^2}{G(1.578\times10^8\text{s})^2}\right)(1.5\times10^{12}\text{m})^3 = \boxed{8.02\times10^{31}\text{kg}}$$